The Complexity of A Class of Linear Equations with Min and Max Operators

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Outline for section 1

Linear equations with min and max operators

Consider $(n_1, n_2, n, \mathcal{J}, \mathbf{q}, \mathbf{b})$ such that the following conditions hold:
(a) $n_1, n_2, n \in \mathbb{N}$,
(b) $n \geq n_1 + n_2$,
(c) $\varnothing \subsetneq \mathcal{J}(i) \subseteq [n]$ for $i \in [n_1 + n_2]$,
(d) $\mathbf{q}_k \in \mathbb{R}^n$ for $n_1 + n_2 < k \leq n$, and
(e) $\mathbf{b} = [0, \ldots, 0, b_{n_1+n_2+1}, \ldots, b_n]^\mathsf{T} \in \mathbb{R}^n$.

A vector $\mathbf{x} = [x_1, \ldots, x_n]^\mathsf{T} \in \mathbb{R}^n$ that satisfies the following system of linear equations with min and max operators is called feasible:

$$\begin{cases} x_i = \min_{l \in \mathcal{J}(i)} x_l, & 1 \leq i \leq n_1, & \text{(1a)} \\ x_j = \max_{l \in \mathcal{J}(j)} x_l, & n_1 < j \leq n_1 + n_2, & \text{(1b)} \\ x_k = \mathbf{q}_k^\mathsf{T} \mathbf{x} + b_k, & n_1 + n_2 < k \leq n. & \text{(1c)} \end{cases}$$

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Notations

Let the indicator vector

$$\mathbf{e}_i := [\delta_{i,1}, \ldots, \delta_{i,n}]^\mathsf{T},\ i \in [n]\,,$$

where $\delta_{i,j}$ is 1 if $i = j$ and 0 otherwise.
Denote

$$\mathcal{Q} := \left\{ [\mathbf{e}_{\ell(1)}, \ldots, \mathbf{e}_{\ell(n_1+n_2)}, \mathbf{q}_{n_1+n_2+1}, \ldots, \mathbf{q}_n]^\mathsf{T} \,\middle|\, \ell(j) \in \mathcal{J}(j) \text{ for all } j \in [n_1 + n_2] \right\}.$$

The convex hull of $\mathcal{Q}$ is denoted as

$$\mathbf{conv}(\mathcal{Q}) := \left\{ \sum_{i \in \mathcal{I}} \alpha_i \mathbf{Q}_i \,\middle|\, \sum_{i \in \mathcal{I}} \alpha_i = 1 \text{ and } \forall\ i \in \mathcal{I},\ \mathbf{Q}_i \in \mathcal{Q} \wedge \alpha_i \geq 0 \right\}.$$

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Restrictive conditions

Condition 1 (Halting or stability)

For all $\mathbf{Q} \in \mathbf{conv}(\mathcal{Q})$, $\lim_{m\to\infty} \mathbf{Q}^m = \mathbf{0}_{n\times n}$.

Condition 2 (Non-negative coefficients)

For all $n_1 + n_2 < k \leq n$, we have that $\mathbf{q}_k \geq 0$ and $b_k \geq 0$.

Condition 3 (Sum up to 1)

For all $n_1 + n_2 < k \leq n$, we have that $\mathbf{q}_k^\top \mathbf{1} + b_k \leq 1$.

Condition 4 (Max-only or min-only)

Either $n_1 = 0$ or $n_2 = 0$.

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Motivating example: verification of neural networks

Given a multi-layer neural network, we are to decide whether the output satisfies

$$x^{\text{out}} < \beta \text{ for input } \mathbf{x}^{\text{in}} \in [0,1]^d.$$

The above problem can be posed as the following system:

$$\begin{cases} \mathbf{x}^{\text{in}} = \max\{0, \min\{\mathbf{x}^{\text{in}}, 1\}\}\,, \\ \mathbf{x}^{\text{Layer 1}} = f(\mathbf{Q}^{\text{Layer 1}}\mathbf{x}^{\text{in}} + \mathbf{b}^{\text{Layer 1}})\,, \\ \vdots \\ \mathbf{x}^{\text{Layer } n} = f(\mathbf{Q}^{\text{Layer } n}\mathbf{x}^{\text{Layer } n-1} + \mathbf{b}^{\text{Layer } n})\,, \\ x^{\text{out}} = {\mathbf{q}^{\text{out}}}^{\mathsf{T}}\, \mathbf{x}^{\text{Layer } n} + b^{\text{out}}\,, \end{cases}$$

where f can be any piecewise linear activation function (*e.g.*, ReLU or Maxout).

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Motivating example: co-evolution

Let $\mathbf{x} \in \mathbb{R}^k_{\geq 0}$ be the current population, and the population of the next generation is given by

$$\mathbf{x}' = \max\left\{\mathbf{Q}\mathbf{x} + \mathbf{b}, 0\right\},$$

in which $\mathbf{Q} \in \mathbb{R}^{k \times k}$ models the internal interactions and $\mathbf{b} \in \mathbb{R}^k$ models the external interventions.

Hence the stable distribution of the ecosystem can be posed as the solution of the following system:

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Motivating example: capital preservation

Consider a cyclic evolution with T periods, each period corresponds to an opportunity for the controller to affect the evolution of the assets, and for the market to affect the assets.

$$\begin{cases} x_{1,1} = \max\{q^1_{1,i}\, x_{T,2} + b^1_{1,i} \mid i \in [m]\}\,, \\ x_{1,2} = \min\{q^1_{2,j}\, x_{1,1} + b^1_{2,j} \mid j \in [\ell]\}\,, \\ \quad \vdots \\ x_{T,1} = \max\{q^T_{1,i}\, x_{T-1,2} + b^T_{1,i} \mid i \in [m]\}\,, \\ x_{T,2} = \min\{q^T_{2,j}\, x_{T,1} + b^T_{2,j} \mid j \in [\ell]\}\,. \end{cases}$$

Outline for section 2

Preliminary

Definition 1 (Decision problem)

The decision problem is: given a system with $(n_1, n_2, n, \mathcal{J}, \mathbf{q}, \mathbf{b})$, an index $i \in [n]$, and a threshold $\beta \in \mathbb{R}$, determine whether there exists a feasible solution $\mathbf{x}$ *s.t.* $x_i < \beta$.

Proposition 1 (Known results)

1. *If $X \subseteq Y$ represents two subsets of conditions, then the decision problem under X is no easier than the decision problem under Y.*
2. *The decision problem without any restriction is in* NP.
3. *The decision problem under* $\{\mathrm{C2,C3}\}$ *is (polynomially) equivalent to simple stochastic games (SSGs), and in* $\mathrm{UP} \cap \mathrm{coUP}$.
4. *The decision problems under* $\{\mathrm{C1,C2,C3}\}$ *and under* $\{\mathrm{C2,C3}\}$ *are equivalent.*
5. *The decision problem under* $\{\mathrm{C2,C3,C4}\}$ *is equivalent to MDPs, and is in* PTIME.

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Complexity lower bounds

Lemma 1

*The decision problem under $\{$**C2,C4**$\}$ is NP-hard.*

Lemma 2

*The decision problem under $\{$**C3,C4**$\}$ is NP-hard.*

Theorem 1

*The decision problems under conditions $\{$**C2,C4**$\}$, $\{$**C3,C4**$\}$, $\{$C2$\}$, $\{$C3$\}$, $\{$C4$\}$, or $\varnothing$ are all NP-complete.*

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Complexity upper bounds

Theorem 2 (Our main result)

The decision problem under $\{$***C1***$\}$ *is in* $\mathrm{UP} \cap \mathrm{coUP}$.

Corollary 1

The decision problems under $\{$***C1,C3,C4***$\}$, $\{$***C1,C3***$\}$, $\{$***C1,C4***$\}$, $\{$***C1***$\}$, *or* $\{$***C1,C2***$\}$ *are in* $\mathrm{UP} \cap \mathrm{coUP}$.

Theorem 3

The decision problem under $\{$***C1,C2,C4***$\}$ *is in* PTIME.

Corollary 2

The decision problems under $\{$C1,C2,C3,C4$\}$, $\{$C2,C3,C4$\}$, *or* $\{$***C1,C2,C4***$\}$ *are in* PTIME.

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The decision problems under $\{$C1,C2,C3,C4$\}$, $\{$C2,C3,C4$\}$, *or* $\{$***C1,C2,C4***$\}$ *are in* PTIME*.*

Equivalence between problem classes

Lemma 3

There is a polynomial time reduction from a system under {C1} *to a system under* {C1,C3,C4}.

Corollary 3

The decision problems under {C1,C3,C4}, {C1,C3}, {C1,C4}, *or* {C1} *are equivalent and no easier than SSGs.*

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Outline for section 3

Preliminary

Definition 2 (Condition checking problem)

The condition checking problem is: given a system and a subset of conditions, determine whether all the conditions are satisfied.

Lemma 4 (Basic and previous results)

1. *The condition checking problems for* $\{\mathrm{C2}\}$, $\{\mathrm{C3}\}$, *or* $\{\mathrm{C4}\}$ *are all easily solvable in linear time.*
2. *The condition checking problem for* $\{\mathrm{C1,C2,C3}\}$ *is in* PTIME.
3. *If* $\{\mathrm{C1}\} \subseteq X \subseteq Y$ *represents two subsets of conditions, then the condition checking problem for* X *is no easier than the condition checking problem for* Y.

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The condition checking problem for $\{$***C1,C3,C4***$\}$ *is* coNP*-hard.*

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Outline for section 4

Summary of results

{**C2,C4**}, {**C3,C4**}, {C2}, {C3}, {C4}, ∅	NP-COMPLETE	Theorem 1
{**C1,C3,C4**}, {**C1,C3**}, {**C1,C4**}, {**C1**}	UP ∩ coUP (SSG-hard)	Corollary 1 & 3
{**C1,C2**}		
{C1,C2,C3}, {C2,C3}		[Con92]
{C1,C2,C3,C4}, {C2,C3,C4}, {**C1,C2,C4**}	PTIME	Corollary 2

Table 1: The complexity of the decision problems under all subsets of conditions. The table describes all the results, with our results in bold. Moreover, for the problems in a row there is a polynomial-time equivalence.

{**C1**}, {**C1,C3**}, {**C1,C4**}, {**C1,C3,C4**}	coNP-hard	Corollary 5
{**C1,C2**}, {**C1,C2,C4**}, {C1,C2,C3}, {C1,C2,C3,C4}	PTIME	Corollary 4

Table 2: The complexity of the condition checking problems for all subsets of conditions in the presence of condition C1. In the absence of condition C1, all other conditions can be checked in linear time. Our results are in bold.

Summary of results

{**C2,C4**}, {**C3,C4**}, {C2}, {C3}, {C4}, ∅	NP-COMPLETE	Theorem 1
{**C1,C3,C4**}, {**C1,C3**}, {**C1,C4**}, {**C1**}	UP ∩ coUP (SSG-hard)	Corollary 1 & 3
{**C1,C2**}		
{C1,C2,C3}, {C2,C3}		[Con92]
{C1,C2,C3,C4}, {C2,C3,C4}, {**C1,C2,C4**}	PTIME	Corollary 2

Table 1: The complexity of the decision problems under all subsets of conditions. The table describes all the results, with our results in bold. Moreover, for the problems in a row there is a polynomial-time equivalence.

{**C1**}, {**C1,C3**}, {**C1,C4**}, {**C1,C3,C4**}	coNP-hard	Corollary 5
{**C1,C2**}, {**C1,C2,C4**}, {C1,C2,C3}, {C1,C2,C3,C4}	PTIME	Corollary 4

Table 2: The complexity of the condition checking problems for all subsets of conditions in the presence of condition C1. In the absence of condition C1, all other conditions can be checked in linear time. Our results are in bold.