# Monotone Near-Zero-Sum Games

**Ruichen Luo***
IST Austria
3400 Klosterneuburg, Austria

**Sebastian U. Stich**
CISPA Helmholtz Center
66386 St. Ingbert, Germany

**Krishnendu Chatterjee**
IST Austria
3400 Klosterneuburg, Austria

## Abstract

Zero-sum and non-zero-sum (aka general-sum) games are relevant in a wide range of applications. While general non-zero-sum games are computationally hard, researchers focus on the special class of monotone games for gradient-based algorithms. However, there is a substantial gap between the gradient complexity of monotone zero-sum and monotone general-sum games. Moreover, in many practical scenarios of games the zero-sum assumption needs to be relaxed. To address these issues, we define a new intermediate class of *monotone near-zero-sum games* that contains monotone zero-sum games as a special case. Then, we present a novel algorithm that transforms the near-zero-sum games into a sequence of zero-sum subproblems, improving the gradient-based complexity for the class. Finally, we demonstrate the applicability of this new class to model practical scenarios of games motivated from the literature.

## 1 Introduction

Zero-sum games (also known as strictly competitive games) and their generalization to non-zero-sum games [39, 21, 32] are crucial in domains like economics [39], artificial intelligence [43], and biology in the form of evolutionary game theory [41, 35].

For two-person non-zero-sum games, [6, 4, 28] show that finding the Nash equilibrium is PPAD-hard, which implies that obtaining efficient computational methods is challenging in general. Hence, researchers often focus on special class of games amenable to efficient gradient-based algorithms. In this paper, we focus on the computationally tractable class of *monotone games* with compact convex strategy spaces.[2]

On the theory side, especially when the conditioning of the two players is imbalanced,[3] there is a substantial gap between the gradient complexity of monotone zero-sum games and that of monotone general-sum games: for monotone general-sum games, the seminal work of [31, 37] establishes a gradient complexity; while for monotone zero-sum games, exploiting recent development in minimax optimization a much better complexity result is obtained [17, 14, 3, 36, 15].

On the application side, strictly competitive scenarios, modeled by monotone zero-sum games, are often insufficient. Real-world game settings frequently involve factors such as transaction fees or semi-cooperation, necessitating a relaxation of the zero-sum assumption.

*This work was partially done during RL's stay at CISPA.

[2]The assumption of compact convex strategy spaces are quite common in the study of games. Theoretically, von Neumann's and Sion's minimax theorem [38, 34] both require the strategy spaces to be compact and convex. Practically, for games with discrete action sets, we may consider probability distributions over the action sets and take the expectation of the players. For instance, in classic games like rock-paper-scissors, by considering probability distributions over actions, the unique Nash equilibrium is for both players to play rock/paper/scissors with probability $1/3$ [21].

[3]A detailed discussion of the implications of imbalanced conditioning is included for completeness in Section A.

Preprint. Under review.

Our work aims to bridge the gap between monotone zero-sum games and general-sum games. For this purpose, we introduce a new intermediate class of monotone games, present a novel algorithm for this class, and show the applicability of this new class. In detail:

- **Theoretical motivation** We define a new intermediate class of games called *monotone near-zero-sum games*, characterized by a smoothness parameter $\delta$ describing the game's proximity to a zero-sum game. This new class of games presents a natural interpolation between monotone zero-sum games and a class of monotone general-sum games based on the near-zero-sum parameter $\delta$, and thus, it partially bridges the gap of the monotone zero-sum and general-sum classes.
- **Main theoretical result** We propose a novel algorithm, Iterative Coupling Linearization (ICL), that converges to an $\varepsilon$-Nash equilibrium within $\widetilde{\mathcal{O}}\left(\left(\frac{L}{\sqrt{\mu\nu}} + \frac{L}{\min\{\mu,\nu\}} \cdot \min\left\{1, \sqrt{\frac{\delta}{\mu+\nu}}\right\}\right) \cdot \log^2\left(\frac{D^2}{\varepsilon}\right)\right)$ gradient queries, where $L$ is the smoothness parameter, $\mu$ and $\nu$ are the strong convexity parameters of the two players, $\delta$ is the near-zero-sum parameter, and $D$ is the diameter. When $\delta$ is sufficiently near-zero and the conditioning of $\mu$, $\nu$ is imbalanced, our algorithm requires fewer gradient queries than existing variational inequality methods for monotone non-zero-sum games [37].
- **Practical applications** Besides the theoretical motivation, we demonstrate the practical relevance of this new class of games. We apply our algorithm to *regularized matrix games* and *competitive games with small additional incentives*, where provably better rates are achieved. Even in the well-studied context of matrix games, this acceleration from the near-zero-sum structure *and* imbalanced conditioning is a new result, to the best of our knowledge.

## 2 Related work

We discuss other classes of games in the literature that bridge the gap between zero-sum and general-sum games.

**Near network zero-sum games** Near network zero-sum games [8] define a class of games that is close to network zero-sum games in terms of maximum pairwise difference [2, 8]. Limited to the setting of two-person games, monotone near-zero-sum games considered in this paper differ in three aspects: (i) The utility functions in this paper can be general functions, rather than bilinear functions; (ii) the difference between near-zero-sum games and zero-sum games in this paper is characterized by (higher-order) smoothness parameter, rather than by function values; and (iii) the solution of near network zero-sum games is taken directly from the zero-sum case, which only guarantees convergence to a neighborhood of the Nash equilibrium.

**Rank-$k$ games** In the setting of matrix games, one of the most significant attempts on bridging the gap between zero-sum and non-zero-sum games is the class of Rank-$k$ games introduced in [12]. As a generalization of zero-sum matrix games, [12] study matrix games where $\mathrm{rank}(\mathbf{A}+\mathbf{B}) = k$, where $\mathbf{A}$ and $\mathbf{B}$ are the payoff matrices of the two players. To find an approximate Nash equilibrium, an FPTAS exists when $k$ is small [12]; to find an exact Nash equilibrium, Rank-1 games can be solved in polynomial time [1], while Rank-3 games are already PPAD-hard [18]. It is crucial to emphasize that monotone near-zero-sum games, as considered in this paper, are fundamentally distinct from Rank-$k$ games. Specifically: (i) The utility functions in this paper can be general functions, rather than bilinear functions; (ii) matrix games can be sufficiently near-zero-sum but still have full rank; and (iii) the focus of this paper is on gradient-based algorithms and complexity within the Nemirovsky-Yudin optimization model [25], while the study of Rank-$k$ games focuses on algorithms and complexity on Turing machines.

## 3 Definitions, previous results, and the new problem class

### 3.1 Basic definitions

This paper studies the *Nash Equilibrium Problem* (NEP) for two-person general-sum games, in which Player 1 wants to maximize its utility function $u_1(\mathbf{x}, \mathbf{y})$ over $\mathbf{x} \in X$ and Player 2 wants to maximize its utility function $u_2(\mathbf{x}, \mathbf{y})$ over $\mathbf{y} \in Y$. Here, $X$ and $Y$ are compact and convex sets, and $u_1(\cdot, \cdot)\colon X \times Y \to \mathbb{R}$ and $u_2(\cdot, \cdot)\colon X \times Y \to \mathbb{R}$ are smooth functions. A pair of decisions

$(\mathbf{x}^*, \mathbf{y}^*) \in X \times Y$ is a *Nash equilibrium* if

$$u_1(\mathbf{x}^*, \mathbf{y}^*) \geq u_1(\mathbf{x}, \mathbf{y}^*), \text{ for all } \mathbf{x} \in X, \text{ and } u_2(\mathbf{x}^*, \mathbf{y}^*) \geq u_2(\mathbf{x}^*, \mathbf{y}), \text{ for all } \mathbf{y} \in Y.$$

A pair of decisions $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in X \times Y$ is an *$\varepsilon$-accurate Nash equilibrium* if there exists a Nash equilibrium $(\mathbf{x}^*, \mathbf{y}^*)$ such that $\|\bar{\mathbf{x}} - \mathbf{x}^*\|^2 + \|\bar{\mathbf{y}} - \mathbf{y}^*\|^2 \leq \varepsilon$. A pair of decisions $(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \in X \times Y$ is an *$\varepsilon$-approximate Nash equilibrium*, if $u_1(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \geq u_1(\mathbf{x}, \hat{\mathbf{y}}) - \varepsilon$ for all $\mathbf{x} \in X$ and $u_2(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \geq u_2(\hat{\mathbf{x}}, \mathbf{y}) - \varepsilon$ for all $\mathbf{y} \in Y$. The relation between accurate and approximate Nash equilibria is discussed in Section B.

The goal of this paper is to find an $\varepsilon$-accurate (or an $\varepsilon$-approximate) Nash equilibrium by iterative algorithms which subsequently query the gradients of the utility functions. To avoid ambiguity, when using the term *gradient complexity* of an NEP, we are referring to the number of gradient queries needed to find an $\varepsilon$-accurate Nash equilibrium.

**Notations** Let $\mathcal{X}$ and $\mathcal{Y}$ be Euclidean spaces. In the space $\mathcal{X} \times \mathcal{Y}$, for all $\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in \mathcal{X} \times \mathcal{Y}$ and $\mathbf{z}' = (\mathbf{x}', \mathbf{y}') \in \mathcal{X} \times \mathcal{Y}$, define $\langle \mathbf{z}', \mathbf{z} \rangle \stackrel{\text{def}}{=} \langle \mathbf{x}', \mathbf{x} \rangle + \langle \mathbf{y}', \mathbf{y} \rangle$. For all these spaces, the norms are those induced by inner products. Assume that the diameter of $X \subseteq \mathcal{X}$ is bounded by $D_X$ and the diameter of $Y \subseteq \mathcal{Y}$ is bounded by $D_Y$. Let $D = \sqrt{D_X^2 + D_Y^2}$. Assume that $u_1(\cdot, \cdot)$ and $u_2(\cdot, \cdot)$ are $L$-smooth, that is,

$$\|\nabla u_1(\mathbf{z}') - \nabla u_1(\mathbf{z})\| \leq L \|\mathbf{z}' - \mathbf{z}\|, \; \|\nabla u_2(\mathbf{z}') - \nabla u_2(\mathbf{z})\| \leq L \|\mathbf{z}' - \mathbf{z}\|, \text{ for all } \mathbf{z}, \mathbf{z}' \in X \times Y.$$

To facilitate our analysis, we adopt the following formulation that decomposes the game into a coupling part and a zero-sum part. Denote

$$g = -\frac{1}{2}(u_1 + u_2), \quad h = \frac{1}{2}(-u_1 + u_2), \quad \mathcal{H} = (\nabla_{\mathbf{x}} h, -\nabla_{\mathbf{y}} h), \quad \mathcal{F} = -(\nabla_{\mathbf{x}} u_1, \nabla_{\mathbf{y}} u_2),$$

where $g$ is the coupling part, $h$ is the zero-sum part, $\mathcal{H}$ is the operator corresponding to the zero-sum part $h$, and $\mathcal{F}$ is the operator corresponding to the game. Then, we have

$$u_1 = -g - h, \quad u_2 = -g + h, \quad \mathcal{F} = \nabla g + \mathcal{H}.$$

Since the utilities $u_1$ and $u_2$ are both $L$-smooth, we have the functions $g$ and $h$ are both $L$-smooth, and the operators $\mathcal{H}$ and $\mathcal{F}$ are both Lipschitz continuous. While similar decompositions can be found in the literature of variational inequalities and game theory [23, 7, 5, 9], we emphasize that this notation is particularly suited to characterize our near-zero-sum games (to be defined later) for explicitly separating the non-zero-sum coupling part.

### 3.2 Problem classes

**General-sum games and PPAD-hardness** The seminal work of [32] establishes the existence of Nash equilibrium for concave games, where $u_1(\cdot, \mathbf{y})$ is concave for any fixed $\mathbf{y} \in Y$ and $u_2(\mathbf{x}, \cdot)$ is concave for any fixed $\mathbf{x} \in X$. Therefore, in this context, computing a Nash equilibrium, or an $\varepsilon$-accurate Nash equilibrium, is a well-defined problem. However, without further restriction, the NEP for concave general-sum games is known to be PPAD-hard [4, 28].

**Monotone general-sum games** To obtain a tractable class of NEPs, we consider further restrictions. Specifically, we consider the following assumptions:

**Assumption 1** (Convex-concave zero-sum part)**.** *There exists $\mu, \nu \in [0, L]$ such that the function $h(\mathbf{x}, \mathbf{y}) - \frac{\mu}{2}\|\mathbf{x}\|^2$ is convex in $\mathbf{x}$ for any fixed $\mathbf{y} \in Y$, and the function $h(\mathbf{x}, \mathbf{y}) - \frac{\nu}{2}\|\mathbf{y}\|^2$ is concave in $\mathbf{y}$ for any fixed $\mathbf{x} \in X$.*

**Assumption 2** (jointly convex coupling part)**.** *The function $g(\cdot, \cdot)$ is jointly convex.*

The operator $\mathcal{H} = (\nabla_{\mathbf{x}} h, -\nabla_{\mathbf{y}} h)$ is monotone with moduli $\min\{\mu, \nu\}$ under Assumption 1, and operator $\nabla g$ is monotone under Assumption 2. Hence, under Assumptions 1 and 2, the game (or the operator $\mathcal{F} = \nabla g + \mathcal{H}$) is monotone with moduli $\min\{\mu, \nu\}$ under Assumption 1 [23], that is,

$$\langle \mathcal{F}(\mathbf{z}') - \mathcal{F}(\mathbf{z}), \mathbf{z}' - \mathbf{z} \rangle \geq \min\{\mu, \nu\} \cdot \|\mathbf{z}' - \mathbf{z}\|^2, \text{ for all } \mathbf{z}, \mathbf{z}' \in X \times Y.$$

In this paper, we refer to a game as a *monotone (general-sum) game* if it satisfies Assumptions 1 and 2. We refer to a game as a *strongly monotone* game if it is a monotone game with modulus $\mu, \nu > 0$. It is known that there exists a unique Nash equilibrium for strongly monotone games [32].

**Monotone zero-sum games (convex-concave minimax optimization)** We now consider a more restrictive problem subclass: monotone zero-sum games. A two-person game is *zero-sum* if $g = 0$. A game is said to be a *monotone zero-sum game* if it is zero-sum and satisfies Assumption 1. Note that monotone zero-sum games trivially satisfy Assumption 2 (since $g = 0$ is convex), and therefore form a subclass of monotone general-sum games. By Sion's minimax theorem [34], the NEP for monotone zero-sum games is equivalent to *convex-concave minimax optimization*, that is finding or approaching a saddle point of the function $h(\cdot, \cdot)$:

$$\min_{\mathbf{x} \in X} \max_{\mathbf{y} \in Y} h(\mathbf{x}, \mathbf{y}) .$$

### 3.3 Previous results

We outline the gradient complexity results of the aforementioned tractable classes of NEPs. To simplify the presentation, we assume for now in Section 3.3 that the modulus $\mu, \nu > 0$ in Assumption 1.

For monotone general-sum games, the NEP can be solved using variational inequality methods for the operator $\mathcal{F}$, leading to the following gradient complexity:

**Proposition 1** ([37])**.** *For monotone general-sum games, an $\varepsilon$-accurate Nash equilibrium can be found with the number of gradient queries bounded by*

$$\mathcal{O}\left(\frac{L}{\min\{\mu, \nu\}} \cdot \log\left(\frac{D^2}{\varepsilon}\right)\right) .$$

For monotone zero-sum games, the gradient complexity can be further improved due to recent advances in minimax optimization.

**Proposition 2** ([14, 3, 36, 15, 44])**.** *For monotone zero-sum games, an $\varepsilon$-accurate Nash equilibrium can be found with the number of gradient queries bounded by*

$$\mathcal{O}\left(\frac{L}{\sqrt{\mu\nu}} \cdot \log\left(\frac{D^2}{\varepsilon}\right)\right) .$$

*This rate is minimax optimal, as $\Omega\left(\frac{L}{\sqrt{\mu\nu}} \cdot \log\left(\frac{D^2}{\varepsilon}\right)\right)$ gradient queries are required in general.*

### 3.4 The new problem class

**Theoretical motivation** As shown in Propositions 1 and 2, a significant gap exists in the gradient complexities for solving NEPs in monotone general-sum games versus monotone zero-sum games, particularly when the conditioning is imbalanced (that is, $\min\{\mu, \nu\} = o(\max\{\mu, \nu\})$). This disparity motivates the exploration of an intermediate problem class that partially bridges this gap.

**Monotone near-zero-sum games** We introduce the class of *monotone $\delta$-near-zero-sum games*, which naturally interpolates between monotone zero-sum games ($\delta = 0$)[4] and monotone general-sum games ($\delta = L$).

**Assumption 3** (Near-zero-sum)**.** *There exists $\delta \in [0, L]$ such that the function $g(\cdot, \cdot)$ is $\delta$-smooth.*

> **Definition** (Monotone Near-Zero-Sum Games)**.** *If a two-person general-sum game satisfies Assumptions 1 to 3, we call it a* monotone $\delta$-near-zero-sum game.

## 4 Algorithm and convergence analysis

We first focus on the algorithm and analysis for strongly monotone near-zero-sum games (that is, the modulus $\mu, \nu > 0$) in Sections 4.1 and 4.2. Then, in Section 4.3, we also present the results for (non-strongly) monotone near-zero-sum games.

[4] In 0-near-zero-sum game, let Player 1 maximize $\mathbf{a}_1^\top \mathbf{x} + \mathbf{b}_1^\top \mathbf{y} - h(\mathbf{x}, \mathbf{y})$ and Player 2 maximize $\mathbf{a}_2^\top \mathbf{x} + \mathbf{b}_2^\top \mathbf{y} + h(\mathbf{x}, \mathbf{y})$, respectively. The Nash equilibrium in the above game is the same as that in the following zero-sum game: Player 1 maximizes $\mathbf{a}_1^\top \mathbf{x} - \mathbf{b}_2^\top \mathbf{y} - h(\mathbf{x}, \mathbf{y})$ and Player 2 maximizes $-\mathbf{a}_1^\top \mathbf{x} + \mathbf{b}_2^\top \mathbf{y} + h(\mathbf{x}, \mathbf{y})$.

**Algorithm 1** Iterative Coupling Linearization (ICL)

**Require:** $\mathbf{z}_0 = (\mathbf{x}_0, \mathbf{y}_0) \in X \times Y$.

1: **for** $t = 0, 1, \cdots, T-1$ **do**

2: Let $\varphi_t(\mathbf{x}, \mathbf{y}) \stackrel{\text{def}}{=}$

$$\langle \nabla_{\mathbf{x}} g(\mathbf{x}_t, \mathbf{y}_t), \mathbf{x} \rangle + \frac{1}{2\eta_t} \|\mathbf{x} - \mathbf{x}_t\|^2 + h(\mathbf{x}, \mathbf{y}) - \langle \nabla_{\mathbf{y}} g(\mathbf{x}_t, \mathbf{y}_t), \mathbf{y} \rangle - \frac{1}{2\eta_t} \|\mathbf{y} - \mathbf{y}_t\|^2 .$$

3: Find an inexact solution $\mathbf{z}_{t+1} = (\mathbf{x}_{t+1}, \mathbf{y}_{t+1}) \in X \times Y$ to $\min_{\mathbf{x} \in X} \max_{\mathbf{y} \in Y} \varphi_t(\mathbf{x}, \mathbf{y})$ such that

$$\langle \nabla_{\mathbf{x}} \varphi_t(\mathbf{z}_{t+1}), \mathbf{x}_{t+1} - \mathbf{x} \rangle - \langle \nabla_{\mathbf{y}} \varphi_t(\mathbf{z}_{t+1}), \mathbf{y}_{t+1} - \mathbf{y} \rangle \le \varepsilon_t, \text{ for all } \mathbf{x} \in X, \ \mathbf{y} \in Y . \tag{1}$$

4: **end for**

### 4.1 Algorithm

While smoothing techniques [26, 16, 3] can achieve a fast $\mathcal{O}\left(\frac{L}{\sqrt{\mu\nu}} \log\left(\frac{D^2}{\varepsilon}\right)\right)$ convergence rate in minimax optimization [17, 14, 3, 15], their direct application to NEPs for non-zero-sum games is complicated by the fact that the smooth minimization transforms the problem into a Stackelberg game, whose solution deviates significantly from a Nash equilibrium (see Section C for more details). Thus, we are not aware of how to apply smoothing directly to general-sum games.

This raises the challenge: *can we leverage the off-the-shelf algorithms designed for zero-sum games to solve the non-zero-sum problems of interest?* Now, we introduce our novel algorithm, Iterative Coupling Linearization (ICL), which overcomes the aforementioned challenge and presents a clean framework to solve non-zero-sum games by using any zero-sum algorithm as an oracle.

**Potential function** Our ICL algorithm leverages the potential function $\Delta\colon X \times Y \to \mathbb{R}$ defined as:

$$\Delta(\mathbf{z}) = \max_{\widetilde{\mathbf{z}} = (\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}) \in X \times Y} \underbrace{g(\mathbf{z}) - g(\widetilde{\mathbf{z}})}_{\text{jointly convex coupling}} + \underbrace{h(\mathbf{x}, \widetilde{\mathbf{y}}) - h(\widetilde{\mathbf{x}}, \mathbf{y})}_{\text{convex-concave zero-sum}}, \text{ for all } \mathbf{z} = (\mathbf{x}, \mathbf{y}) \in X \times Y .$$

This potential function decomposes into a jointly convex coupling part and a convex-concave zero-sum part. We show below in Propositions 3 and 4 that minimizing our potential function $\Delta(\cdot)$ is sufficient for finding a Nash equilibrium (with detailed proofs in Section D.1):

**Proposition 3.** *For any* $\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in X \times Y$*, we have* $\Delta(\mathbf{z}) \ge 0$ *and*

$$2\Delta(\mathbf{z}) \ge \max_{\widetilde{\mathbf{z}} = (\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}) \in X \times Y} u_1(\widetilde{\mathbf{x}}, \mathbf{y}) - u_1(\mathbf{x}, \mathbf{y}) + u_2(\mathbf{x}, \widetilde{\mathbf{y}}) - u_2(\mathbf{x}, \mathbf{y}) .$$

**Proposition 4.** *Let* $\mathbf{z}^* \in X \times Y$*. In monotone general-sum games,* $\mathbf{z}^*$ *is the Nash equilibrium if and only if* $\Delta(\mathbf{z}^*) = 0$*.*

**Algorithm description** Our ICL algorithm solves the monotone near-zero-sum game by iteratively linearizing the jointly convex coupling part, thereby transforming the non-zero-sum game into a sequence of zero-sum subproblems. The pseudocode is presented in Algorithm 1. Specifically, at iteration $t$, we obtain $\mathbf{z}_{t+1} = (\mathbf{x}_{t+1}, \mathbf{y}_{t+1})$ by solving the following minimax optimization inexactly:

$$\min_{\mathbf{x} \in X} \max_{\mathbf{y} \in Y} \langle \nabla_{\mathbf{x}} g(\mathbf{z}_t), \mathbf{x} \rangle + \frac{1}{2\eta_t} \|\mathbf{x} - \mathbf{x}_t\|^2 + h(\mathbf{x}, \mathbf{y}) - \langle \nabla_{\mathbf{y}} g(\mathbf{z}_t), \mathbf{y} \rangle - \frac{1}{2\eta_t} \|\mathbf{y} - \mathbf{y}_t\|^2 ,$$

with the inexactness condition stated as in Equation (1) in Algorithm 1.

**Novelty** Our key novelty and technical contributions lie in (a) the design of potential function; (b) the outer loop of the ICL algorithm; and (c) the derivation of the descent lemma (see Lemma 5). Together, these enable us to *solve non-zero-sum problems by using any off-the-shelf zero-sum algorithm as an oracle*, with natural intuition and concise proofs.

### 4.2 Convergence analysis

Throughout Section 4.2, let $\mathbf{z}^* = (\mathbf{x}^*, \mathbf{y}^*)$ be the (unique) Nash equilibrium for the game. We only present the main proof ideas in this section, and the detailed proofs can be found in Section D.2.

The core of our convergence analysis is to use the properties of the potential function and derive the following descent lemma, which characterizes the progress made in each outer loop iteration of Algorithm 1.

**Lemma 5** (Descent lemma)**.** *In monotone $\delta$-near-zero-sum games, for $\eta_t \le \frac{1}{\delta}$, Algorithm 1 ensures*

$$\left(\frac{1}{2\eta_t} + \frac{\min\{\mu,\nu\}}{2}\right)\|\mathbf{z}_{t+1} - \mathbf{z}^*\|^2 \le \frac{1}{2\eta_t}\|\mathbf{z}_t - \mathbf{z}^*\|^2 + \varepsilon_t\,.$$

*Proof Sketch.* By Assumption 1, we can upper bound the convex-concave zero-sum part

$$h(\mathbf{x}_{t+1},\mathbf{y}^*) - h(\mathbf{x}^*,\mathbf{y}_{t+1}) \le \langle \mathcal{H}(\mathbf{z}_{t+1}), \mathbf{z}_{t+1} - \mathbf{z}^*\rangle - \frac{\mu}{2}\|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2 - \frac{\nu}{2}\|\mathbf{y}_{t+1} - \mathbf{y}^*\|^2\,. \tag{2}$$

By Assumptions 2 and 3, we can upper bound the jointly convex coupling part

$$g(\mathbf{z}_{t+1}) - g(\mathbf{z}^*) \le \langle \nabla g(\mathbf{z}_t), \mathbf{z}_{t+1} - \mathbf{z}^*\rangle + \frac{\delta}{2}\|\mathbf{z}_{t+1} - \mathbf{z}_t\|^2\,. \tag{3}$$

In view of

$$\frac{1}{2}\langle \mathbf{z}_{t+1} - \mathbf{z}_t, \mathbf{z}_{t+1} - \mathbf{z}^*\rangle = \|\mathbf{z}_{t+1} - \mathbf{z}^*\|^2 - \|\mathbf{z}_t - \mathbf{z}^*\|^2 + \|\mathbf{z}_{t+1} - \mathbf{z}_t\|^2\,, \tag{4}$$

we have

$$\begin{aligned}
0 &= -\Delta(\mathbf{z}^*) \le g(\mathbf{z}_{t+1}) - g(\mathbf{z}^*) + h(\mathbf{x}_{t+1},\mathbf{y}^*) - h(\mathbf{x}^*,\mathbf{y}_{t+1}) \\
&\overset{(2)(3)}{\le} \langle \nabla g(\mathbf{z}_t) + \mathcal{H}(\mathbf{z}_{t+1}), \mathbf{z}_{t+1} - \mathbf{z}^*\rangle - \frac{\mu}{2}\|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2 - \frac{\nu}{2}\|\mathbf{y}_{t+1} - \mathbf{y}^*\|^2 + \frac{\delta}{2}\|\mathbf{z}_{t+1} - \mathbf{z}_t\|^2 \\
&\le \left\langle \nabla g(\mathbf{z}_t) + \mathcal{H}(\mathbf{z}_{t+1}) + \frac{1}{\eta_t}(\mathbf{z}_{t+1} - \mathbf{z}_t), \mathbf{z}_{t+1} - \mathbf{z}^*\right\rangle - \frac{1}{\eta_t}\langle \mathbf{z}_{t+1} - \mathbf{z}_t, \mathbf{z}_{t+1} - \mathbf{z}^*\rangle \\
&\quad - \frac{\min\{\mu,\nu\}}{2}\|\mathbf{z}_{t+1} - \mathbf{z}^*\|^2 + \frac{\delta}{2}\|\mathbf{z}_{t+1} - \mathbf{z}_t\|^2 \\
&\overset{(1)(4)}{\le} \varepsilon_t + \frac{1}{2\eta_t}\|\mathbf{z}_t - \mathbf{z}^*\|^2 - \left(\frac{1}{2\eta_t} + \frac{\min\{\mu,\nu\}}{2}\right)\|\mathbf{z}_{t+1} - \mathbf{z}^*\|^2 - \left(\frac{1}{2\eta_t} - \frac{\delta}{2}\right)\|\mathbf{z}_{t+1} - \mathbf{z}_t\|^2\,,
\end{aligned}$$

where the first equality follows from Proposition 4 and the first inequality follows from the definition of $\Delta(\cdot)$. Finally, the desired bound follows from $\eta_t \le \frac{1}{\delta}$. □

Building upon the descent lemma, we can establish the convergence rate of the outer loop.

**Lemma 6** (Outer loop convergence)**.** *Let $\eta_t = \eta \in (0, \frac{1}{\delta}]$, for all $t \in [0, T-1] \cap \mathbb{Z}$. Denote $\theta = \frac{\min\{\mu,\nu\}}{\eta^{-1} + \min\{\mu,\nu\}}$. Let $\varepsilon_t \le \frac{\theta\varepsilon}{4\eta}$, for all $t \in [0, T-1] \cap \mathbb{Z}$. For strongly monotone $\delta$-near-zero-sum games, if the outer loop iterate $t \ge \frac{1}{\theta}\log\frac{2D^2}{\varepsilon}$, then Algorithm 1 converges to an $\varepsilon$-accurate Nash equilibrium, that is, $\|\mathbf{z}_t - \mathbf{z}^*\|^2 \le \varepsilon$.*

For the inner loop, any optimal gradient method from previous work [14, 3, 36, 15] can be used. The gradient complexity of the inner loop is summarized as follows.

**Lemma 7** (Inner loop complexity [14, 3, 36, 15])**.** *Under Assumption 1 with modulus $\mu, \nu > 0$, at each iteration $t \in [0, T-1] \cap \mathbb{Z}$, for $\eta_t \ge \frac{1}{L}$, the inexact solution $(\mathbf{x}_{t+1}, \mathbf{y}_{t+1})$ in Equation* (1) *of Algorithm 1 can be found with the number of gradient queries bounded by $\mathcal{O}\left(\frac{L}{\sqrt{(\eta_t^{-1}+\mu)(\eta_t^{-1}+\nu)}} \cdot \log\left(\frac{LD^2}{\varepsilon_t}\right)\right)$.*

Combining the outer loop convergence result (Lemma 6) with the inner loop complexity (Lemma 7), we obtain the main theoretical result of this paper, the overall gradient complexity of Algorithm 1.

**Theorem 1** (Main theoretical result)**.** *Denote $\eta = \min\left\{\frac{1}{\delta}, \frac{1}{\min\{\mu,\nu\}}\right\}$ and $\theta = \frac{\min\{\mu,\nu\}}{\eta^{-1} + \min\{\mu,\nu\}}$. Let $\eta_t = \eta$ and $\varepsilon_t = \frac{\theta\varepsilon}{4\eta}$, for all $t \in [0, T-1] \cap \mathbb{Z}$. For strongly monotone $\delta$-near-zero-sum games, for $T \ge \frac{1}{\theta}\log\frac{2D^2}{\varepsilon}$, the outer loop iterates of Algorithm 1 converge to an $\varepsilon$-accurate Nash equilibrium with the number of gradient queries bounded by*

$$\mathcal{O}\left(\left(\frac{L}{\sqrt{\mu\nu}} + \frac{L}{\min\{\mu,\nu\}} \cdot \min\left\{1, \sqrt{\frac{\delta}{\mu+\nu}}\right\}\right) \cdot \log\left(\frac{LD^2}{\min\{\mu,\nu\} \cdot \varepsilon}\right)\log\left(\frac{D^2}{\varepsilon}\right)\right)\,.$$

Finally, we highlight the conditions under which Algorithm 1 achieves a faster convergence rate compared to variational inequality methods [37].

REMARK 1 (Acceleration in strongly monotone near-zero-sum games with imbalanced conditioning). *Consider the strongly monotone near-zero-sum games where* $\min\{\mu,\nu\} + \delta = o(\max\{\mu,\nu\})$. *The gradient complexity of Algorithm 1 is given by* $\widetilde{\mathcal{O}}\left(\left(\frac{L}{\sqrt{\mu\nu}} + \frac{L}{\min\{\mu,\nu\}}\cdot\sqrt{\frac{\delta}{\mu+\nu}}\right)\cdot\log^2\left(\frac{D^2}{\varepsilon}\right)\right)$ *or, equivalently,* $\widetilde{\mathcal{O}}\left(\left(\frac{L}{\min\{\mu,\nu\}}\cdot\sqrt{\frac{\min\{\mu,\nu\}+\delta}{\mu+\nu}}\right)\cdot\log^2\left(\frac{D^2}{\varepsilon}\right)\right)$,[5] *which (ignoring logarithm terms) improves upon the* $\mathcal{O}\left(\frac{L}{\min\{\mu,\nu\}}\cdot\log\left(\frac{D^2}{\varepsilon}\right)\right)$ *gradient complexity of variational inequality methods as stated in Proposition 1. We also remark that for the special case of zero-sum games (*$\delta = 0$*), our gradient complexity recovers the optimal* $\mathcal{O}\left(\frac{L}{\sqrt{\mu\nu}}\cdot\log\left(\frac{D^2}{\varepsilon}\right)\right)$ *gradient complexity as stated in Proposition 2 up to a logarithm term.*

### 4.3 Acceleration in non-strongly monotone near-zero-sum games

In this section, we present the results for non-strongly monotone near-zero-sum games where the modulus $\mu$ and $\nu$ can be zero. We first state the known result for general-sum games in literature.

**Proposition 8** ([24])**.** *For monotone general-sum games where* $\mu = 0$ *or* $\nu = 0$*, an* $\varepsilon$*-approximate Nash equilibrium can be found within* $\mathcal{O}\left(\frac{LD^2}{\varepsilon}\right)$ *gradient queries.*

Then, we provide our result, which is obtained by a similar reduction as in [17, 40, 36]. The proof can be found in Section D.3.

**Corollary 9.** *For monotone* $\delta$*-near-zero-sum games where* $\mu = 0$ *or* $\nu = 0$*, an* $\varepsilon$*-approximate Nash equilibrium can be found within*

$$\mathcal{O}\left(\left(\frac{L}{\sqrt{\bar{\mu}\bar{\nu}}} + \frac{L}{\min\{\bar{\mu},\bar{\nu}\}}\cdot\min\left\{1,\sqrt{\frac{\delta}{\bar{\mu}+\bar{\nu}}}\right\}\right)\cdot\log\left(\frac{L^2D^2}{\min\{\bar{\mu},\bar{\nu}\}\cdot\varepsilon}\right)\log\left(\frac{LD^2}{\varepsilon}\right)\right)$$

*gradient queries, where* $\bar{\mu} = \mu + \min\left\{\frac{\varepsilon}{2D_X^2}, L\right\}$ *and* $\bar{\nu} = \nu + \min\left\{\frac{\varepsilon}{2D_Y^2}, L\right\}$.

REMARK 2 (Acceleration in non-strongly monotone near-zero-sum games). *For non-strongly monotone general-sum games, our rate of finding an* $\varepsilon$*-approximate Nash equilibrium (ignoring logarithm terms) is faster than the* $\mathcal{O}\left(\frac{LD^2}{\varepsilon}\right)$ *rate in literature when the conditioning* $\min\{\bar{\mu},\bar{\nu}\} + \delta \ll \max\{\bar{\mu},\bar{\nu}\}$ *holds or when the conditioning* $\frac{\varepsilon}{D^2} \ll \min\{\bar{\mu},\bar{\nu}\}$ *holds.*

## 5 Application examples

In this section, we present practical examples of games that satisfy the conditions outlined in Remark 1. We focus on the application of our approach, while the proof details are presented in Section D.4.

### 5.1 Our approach for regularized matrix games

**Regularized matrix games** We demonstrate the applicability of Iterative Coupling Linearization to regularized matrix games. Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be compact convex sets, and let $\mathbf{A},\mathbf{B} \in \mathbb{R}^{m\times n}$ with $\|\mathbf{A}\| \le L$, $\|\mathbf{B}\| \le L$, and $\left\|\frac{\mathbf{A}+\mathbf{B}}{2}\right\| \le \beta$. Let $\mathcal{R}\colon X\times Y \to \mathbb{R}$ be an $L$-smooth regularizer that is $\mu$-strongly concave-$\nu$-strongly convex. Assume that $\beta \le \frac{1}{2}\sqrt{\mu\nu}$. Let Player 1 maximize

$$u_1(\mathbf{x},\mathbf{y}) = \langle \mathbf{A}\mathbf{x},\mathbf{y}\rangle + \mathcal{R}(\mathbf{x},\mathbf{y})$$

over $\mathbf{x} \in X$ and Player 2 maximize

$$u_2(\mathbf{x},\mathbf{y}) = \langle \mathbf{B}\mathbf{x},\mathbf{y}\rangle - \mathcal{R}(\mathbf{x},\mathbf{y})$$

over $\mathbf{y} \in Y$. Since the game is $\min\{\frac{\mu}{2},\frac{\nu}{2}\}$-strongly monotone, classic variational inequality methods yield an $\varepsilon$-accurate Nash equilibrium within $\mathcal{O}\left(\frac{L}{\min\{\mu,\nu\}}\cdot\log\left(\frac{D^2}{\varepsilon}\right)\right)$ gradient queries [37].

[5] In the $\widetilde{\mathcal{O}}(\cdot)$ notations, the poly-logarithm terms are omitted.

Now, we show that our ICL algorithm can be applied to get a faster rate by leveraging the near-zero-sum structure. Since $-\frac{1}{2}(u_1 + u_2)(\cdot,\cdot)$ is not jointly convex, violating Assumption 2, we first apply a "convex reformulation" technique.

**Convex reformulation technique** Specifically, we choose the parameters $\beta_1$ and $\beta_2$ based on the relationship between $2\beta$, $\mu$, and $\nu$: (i) If $2\beta \le \mu$ and $2\beta \le \nu$, let $\beta_1 = \beta_2 = \beta$; (ii) if $\mu \le 2\beta \le \nu$, let $\beta_1 = \frac{\mu}{2}$ and $\beta_2 = \frac{2\beta^2}{\mu}$; and (iii) if $\nu \le 2\beta \le \mu$, let $\beta_1 = \frac{2\beta^2}{\nu}$ and $\beta_2 = \frac{\nu}{2}$. With these choices, we always have $\beta_1 \le \frac{\mu}{2}$, $\beta_2 \le \frac{\nu}{2}$, and $\sqrt{\beta_1\beta_2} = \beta$.

We then reformulate the problem as follows: Player 1 maximizes

$$\widetilde{u}_1(\mathbf{x},\mathbf{y}) = u_1(\mathbf{x},\mathbf{y}) - \beta_2 \left\|\mathbf{y}\right\|^2$$

over $\mathbf{x} \in X$, and Player 2 maximizes

$$\widetilde{u}_2(\mathbf{x},\mathbf{y}) = u_2(\mathbf{x},\mathbf{y}) - \beta_1 \left\|\mathbf{x}\right\|^2$$

over $\mathbf{y} \in Y$. This reformulated NEP has the same Nash equilibrium as the original. Let

$$\widetilde{g}(\mathbf{x},\mathbf{y}) = -\left\langle \left(\frac{\mathbf{A}+\mathbf{B}}{2}\right)\mathbf{x},\mathbf{y}\right\rangle + \left(\frac{\beta_1}{2}\left\|\mathbf{x}\right\|^2 + \frac{\beta_2}{2}\left\|\mathbf{y}\right\|^2\right)$$

and

$$\widetilde{h}(\mathbf{x},\mathbf{y}) = -\left\langle \left(\frac{\mathbf{A}-\mathbf{B}}{2}\right)\mathbf{x},\mathbf{y}\right\rangle - \left(\mathcal{R}(\mathbf{x},\mathbf{y}) + \frac{\beta_1}{2}\left\|\mathbf{x}\right\|^2 - \frac{\beta_2}{2}\left\|\mathbf{y}\right\|^2\right) .$$

Then, $\widetilde{u}_1 = -\widetilde{g} - \widetilde{h}$ and $\widetilde{u}_2 = -\widetilde{g} + \widetilde{h}$. Since $\beta \le \frac{1}{2}\sqrt{\mu\nu}$, by Cauchy-Schwartz inequality, $\widetilde{g}(\cdot,\cdot)$ is jointly convex. Further, by the choices of $\beta_1$ and $\beta_2$, we have $\widetilde{g}(\cdot,\cdot)$ is $(\beta + \max\{\beta_1,\beta_2\})$-smooth, and $\widetilde{h}(\cdot,\cdot)$ is $\frac{\mu}{2}$-strongly convex-$\frac{\nu}{2}$-strongly concave.

**Our approach applied to reformulated games** Now applying Algorithm 1 to the reformulated NEP, by Theorem 1, we obtain an $\varepsilon$-accurate Nash equilibrium with the number of gradient queries bounded by

$$\widetilde{\mathcal{O}}\left(\left(\frac{L}{\sqrt{\mu\nu}} + \frac{L}{\min\{\mu,\nu\}}\cdot\frac{\beta}{\sqrt{\mu\nu}}\right)\cdot\log^2\left(\frac{D^2}{\varepsilon}\right)\right) .$$

When $\min\{\mu,\nu\} + \beta = o\left(\frac{1}{2}\sqrt{\mu\nu}\right)$, this rate surpasses the best-known $\mathcal{O}\left(\frac{L}{\min\{\mu,\nu\}}\cdot\log\left(\frac{D^2}{\varepsilon}\right)\right)$ gradient complexity of variational inequality methods [37]. This acceleration leveraging the near-zero-sum structure is a new result even in the well-studied context of matrix games, to our knowledge.

EXAMPLE 1 (Matrix games with transaction fees). *Consider regularized matrix games with transaction fees. Let* $X = \mathcal{P}_n \overset{\text{def}}{=} \left\{\mathbf{x} \in \mathbb{R}^n_{\ge 0} \mid x_1 + \cdots + x_n = 1\right\}$ *and* $Y = \mathcal{P}_m \overset{\text{def}}{=} \left\{\mathbf{y} \in \mathbb{R}^m_{\ge 0} \mid y_1 + \cdots + y_m = 1\right\}$. *Let* $\mathbf{M} \in \mathbb{R}^{m\times n}$ *be the payoff matrix of Player 1* without transaction fee*, with* $-\mathbf{M}$ *as the payoff matrix of Player 2* without transaction fee*. Assume* $\|\mathbf{M}\| \le L$. *Denote* $\mathbf{M}_+ \overset{\text{def}}{=} \frac{1}{2}(\mathbf{M} + \text{abs}\,(\mathbf{M}))$ *and* $\mathbf{M}_- \overset{\text{def}}{=} \frac{1}{2}(-\mathbf{M} + \text{abs}\,(\mathbf{M}))$.[6]

*Now, suppose there is a* transaction fee *of* $\rho \in [0,1]$ *charged by some third party on every payment. Then, the payoff matrices of Player 1 and Player 2* with transaction fees *are*

$$\mathbf{A} = (1-\rho)\mathbf{M}_+ - \mathbf{M}_- \quad \textit{and} \quad \mathbf{B} = -\mathbf{M}_+ + (1-\rho)\mathbf{M}_- .$$

*Let* $\mathcal{R}_1\colon X \to \mathbb{R}$ *and* $\mathcal{R}_2\colon Y \to \mathbb{R}$ *be* $L$*-smooth regularizers that are* $\mu$*- and* $\nu$*-strongly concave, respectively. Assume that* $\rho\left\|\text{abs}\,(\mathbf{M})\right\| = o\left(\sqrt{\mu\nu}\right)$.

*Let Player 1 maximize*

$$u_1(\mathbf{x},\mathbf{y}) = \mathcal{R}_1(\mathbf{x}) + \langle \mathbf{A}\mathbf{x},\mathbf{y}\rangle - \mathcal{R}_2(\mathbf{y})$$

*over* $\mathbf{x} \in X$*, and Player 2 maximize*

$$u_2(\mathbf{x},\mathbf{y}) = -\mathcal{R}_1(\mathbf{x}) + \langle \mathbf{B}\mathbf{x},\mathbf{y}\rangle + \mathcal{R}_2(\mathbf{y})$$

*over* $\mathbf{y} \in Y$*. Applying the convex reformulation technique and then Algorithm 1, we obtain an* $\varepsilon$*-accurate Nash equilibrium with the number of gradient queries bounded by*

$$\widetilde{\mathcal{O}}\left(\left(\frac{L}{\sqrt{\mu\nu}} + \frac{L}{\min\{\mu,\nu\}}\cdot\frac{\rho\left\|\text{abs}\,(\mathbf{M})\right\|}{\sqrt{\mu\nu}}\right)\cdot\log^2\left(\frac{D^2}{\varepsilon}\right)\right) .$$

[6] abs $(\mathbf{M})$ represents a matrix of the same dimensions as $\mathbf{M}$ where each element is the absolute value of the corresponding element in $\mathbf{M}$. A simple illustration is given in Section E.

### 5.2 Our approach for competitive games with small additional incentives

We show the applicability of Iterative Coupling Linearization to competitive games with small additional incentives. Let $X$ and $Y$ be compact convex sets in Euclidean spaces. Let $h\colon X \times Y \to \mathbb{R}$ be the competition payoff function, which is $L$-smooth and $\mu$-strongly convex-$\nu$-strongly concave. Let $g\colon X \times Y \to \mathbb{R}$ be the additional incentive function, which is $\beta$-smooth with $\beta \leq L$. Let Player 1 maximize $u_1 = -g - h$ over $\mathbf{x} \in X$, and Player 2 maximize $u_2 = -g + h$ over $\mathbf{y} \in Y$.

We explore two scenarios where the games are $\min\{\frac{\mu}{2}, \frac{\nu}{2}\}$-strongly monotone, to which our ICL algorithm as well as the classic variational inequalities [37] can be applied:

1. If $g(\cdot,\cdot)$ is jointly convex and $\beta = o\left(\max\{\mu,\nu\}\right)$, applying Algorithm 1 directly yields an $\varepsilon$-accurate Nash equilibrium with the number of gradient queries bounded by

$$\widetilde{\mathcal{O}}\left(\left(\frac{L}{\sqrt{\mu\nu}} + \frac{L}{\min\{\mu,\nu\}} \cdot \sqrt{\frac{\beta}{\mu+\nu}}\right) \cdot \log^2\left(\frac{D^2}{\varepsilon}\right)\right).$$

2. If $\beta = o\left(\frac{1}{2}\min\{\mu,\nu\}\right)$, we first apply the "convex reformulation" technique. We reformulate the problem as follows: Player 1 maximizes

$$\widetilde{u}_1(\mathbf{x},\mathbf{y}) = u_1(\mathbf{x},\mathbf{y}) - \beta\,\|\mathbf{y}\|^2$$

over $\mathbf{x} \in X$, and Player 2 maximizes

$$\widetilde{u}_2(\mathbf{x},\mathbf{y}) = u_2(\mathbf{x},\mathbf{y}) - \beta\,\|\mathbf{x}\|^2$$

over $\mathbf{y} \in Y$. This reformulated NEP has the same Nash equilibrium as the original. Let $\widetilde{g} = -\frac{1}{2}(\widetilde{u}_1 + \widetilde{u}_2)$ and $\widetilde{h} = \frac{1}{2}(-\widetilde{u}_1 + \widetilde{u}_2)$ . Then, $\widetilde{h}(\cdot,\cdot)$ is $\frac{\mu}{2}$-strongly convex-$\frac{\nu}{2}$-strongly concave, and $\widetilde{g}(\cdot,\cdot)$ is jointly convex and $2\beta$-smooth.

Applying Algorithm 1 to the reformulated NEP, we obtain an $\varepsilon$-accurate Nash equilibrium with the number of gradient queries bounded by

$$\widetilde{\mathcal{O}}\left(\frac{L}{\sqrt{\mu\nu}} \cdot \log^2\left(\frac{D^2}{\varepsilon}\right)\right).$$

In both scenarios 1 and 2, our gradient queries are fewer than the $\mathcal{O}\left(\frac{L}{\min\{\mu,\nu\}} \cdot \log\left(\frac{D^2}{\varepsilon}\right)\right)$ gradient queries of the classic variational inequality methods [37]. We remark that, in scenario 2, the strict conditioning of $\beta = o\left(\frac{1}{2}\min\{\mu,\nu\}\right)$ is required and is in sharp contrast to the weaker conditioning of $\beta = o\left(\frac{1}{2}\sqrt{\mu\nu}\right)$ in the "convex reformulation" in Section 5.1, where the Cauchy-Schwartz inequality can be used under the bilinear coupling structure.

EXAMPLE 2 (Competitive games with small cooperation incentives). *Consider the games where cooperation coexists with competition. Let $X \subseteq X_\mathrm{a} \times X_\mathrm{b}$ and $Y \subseteq Y_\mathrm{a} \times Y_\mathrm{b}$ be compact convex sets in Euclidean spaces. For $\mathbf{x} = (\mathbf{x}_\mathrm{a}, \mathbf{x}_\mathrm{b}) \in X$, $\mathbf{x}_\mathrm{a} \in X_\mathrm{a}$ represents Player 1's effort in cooperation, and $\mathbf{x}_\mathrm{b} \in X_\mathrm{b}$ represents Player 1's effort in competition (and similarly for $\mathbf{y} = (\mathbf{y}_\mathrm{a}, \mathbf{y}_\mathrm{b}) \in Y$). Let $f_\mathrm{a}\colon X_\mathrm{a} \times Y_\mathrm{a} \to \mathbb{R}$ be the cooperation incentive function given by*

$$f_\mathrm{a}(\mathbf{x}_\mathrm{a}, \mathbf{y}_\mathrm{a}) = \mathcal{R}_1(\mathbf{x}_\mathrm{a}) + \widetilde{g}(\mathbf{x}_\mathrm{a}, \mathbf{y}_\mathrm{a}) + \mathcal{R}_2(\mathbf{y}_\mathrm{a}),$$

*where the regularizer $\mathcal{R}_1\colon X_\mathrm{a} \to \mathbb{R}$ is $\mu$-strongly convex and $L$-smooth, the function $\widetilde{g}\colon X_\mathrm{a} \times Y_\mathrm{a} \to \mathbb{R}$ is jointly convex and $\beta$-smooth, and the regularizer $\mathcal{R}_2\colon Y_\mathrm{a} \to \mathbb{R}$ is $\nu$-strongly convex and $L$-smooth. Let $f_\mathrm{b}\colon X_\mathrm{b} \times Y_\mathrm{b} \to \mathbb{R}$ be the competition payoff function, which is $L$-smooth and $\mu$-strongly convex-$\nu$-strongly concave. Assume that $\beta = o\left(\max\{\mu,\nu\}\right)$.*

*Let Player 1 maximize*

$$u_1(\mathbf{x},\mathbf{y}) = -f_\mathrm{a}(\mathbf{x}_\mathrm{a}, \mathbf{y}_\mathrm{a}) - f_\mathrm{b}(\mathbf{x}_\mathrm{b}, \mathbf{y}_\mathrm{b})$$

*over $\mathbf{x} \in X$, and Player 2 maximize*

$$u_2(\mathbf{x},\mathbf{y}) = -f_\mathrm{a}(\mathbf{x}_\mathrm{a}, \mathbf{y}_\mathrm{a}) + f_\mathrm{b}(\mathbf{x}_\mathrm{b}, \mathbf{y}_\mathrm{b})$$

*over $\mathbf{y} \in Y$. Denoting $\widetilde{h}(\mathbf{x},\mathbf{y}) = \mathcal{R}_1(\mathbf{x}_\mathrm{a}) + f_\mathrm{b}(\mathbf{x}_\mathrm{b}, \mathbf{y}_\mathrm{b}) - \mathcal{R}_2(\mathbf{y}_\mathrm{a})$, the NEP can be reformulated as Player 1 maximizing*

$$\widetilde{u}_1(\mathbf{x},\mathbf{y}) = -\widetilde{g}(\mathbf{x}_\mathrm{a}, \mathbf{y}_\mathrm{a}) - \widetilde{h}(\mathbf{x},\mathbf{y})$$

*over $\mathbf{x} \in X$, and Player 2 maximizing*

$$\widetilde{u}_2(\mathbf{x}, \mathbf{y}) = -\widetilde{g}(\mathbf{x}_a, \mathbf{y}_a) + \widetilde{h}(\mathbf{x}, \mathbf{y})$$

*over $\mathbf{y} \in Y$. Applying Algorithm 1 as detailed above, we obtain an $\varepsilon$-accurate Nash equilibrium with the number of gradient queries bounded by*

$$\widetilde{\mathcal{O}}\left(\left(\frac{L}{\sqrt{\mu\nu}} + \frac{L}{\min\{\mu, \nu\}} \cdot \sqrt{\frac{\beta}{\mu + \nu}}\right) \cdot \log^2\left(\frac{D^2}{\varepsilon}\right)\right).$$

*As a final remark, the modeling of the coexistence of competition and cooperation is by no means new. Early works [20, 22, 33, 30, 11] provided preliminary solutions to semi-cooperative games. More recent developments include the* coco value *[10] and the cooperative equilibrium [7], to name a few. Indeed, these theories are often applied to the scenarios where cooperation dominates, and optimization techniques have been used to accelerate the dominant cooperation part [5]. Our work contributes to this line of research on semi-cooperative games where competition dominates, yet there is a small cooperation incentive.*

## 6 Basic numerical experiments

We conducted numerical experiments to validate our theoretical results, focusing on matrix games with transaction fees as in Example 1. We set $n = m = 10000$, $\mu = 10^{-4}$, $\nu = 1$, and $\varepsilon = 10^{-7}$. A sparse, random matrix $\mathbf{M} \in \mathbb{R}^{m \times n}$ such that $\|\mathbf{M}\| = 1$ was generated. The regularizers were defined as $\mathcal{R}_1(\mathbf{x}) = -\frac{\mu}{2}\|\mathbf{x}\|^2$ and $\mathcal{R}_2(\mathbf{y}) = -\frac{\nu}{2}\|\mathbf{y}\|^2$. We varied the transaction fee $\rho$ from $\{0.00\%, 0.03\%, \cdots, 0.18\%\}$. Our implementation of ICL (Algorithm 1), detailed in Section 5.1, used the Lifted Primal-Dual method [36] for the inner loop. We compared ICL against the Optimistic Gradient Descent Ascent (OGDA) [29] and Extra-Gradient (EG) [13] methods for variational inequalities. More details and additional experiments are provided in Section F, and our code is available in the supplementary material.

Table 1: Gradient query counts (in thousands) to converge to an $\varepsilon$-accurate Nash equilibrium under various transaction fees. Error bars indicate 2-sigma variations across 10 independent runs.

| Methods \ Transaction Fee $\rho$ | 0.00% | 0.03% | 0.06% | 0.09% | 0.12% | 0.15% | 0.18% |
|---|---|---|---|---|---|---|---|
| ICL (Algorithm 1) | $\mathbf{9.1 \pm 0.0}$ | $\mathbf{22.6 \pm 0.4}$ | $\mathbf{42.2 \pm 0.3}$ | $\mathbf{65.0 \pm 0.3}$ | $\mathbf{75.7 \pm 0.3}$ | $113.7 \pm 0.7$ | $123.8 \pm 0.6$ |
| OGDA [29] | $93.9 \pm 0.5$ | $93.9 \pm 0.5$ | $93.9 \pm 0.5$ | $93.9 \pm 0.5$ | $93.9 \pm 0.5$ | $\mathbf{94.0 \pm 0.6}$ | $\mathbf{94.0 \pm 0.6}$ |
| EG [13] | $132.9 \pm 0.8$ | $132.9 \pm 0.8$ | $132.9 \pm 0.8$ | $132.9 \pm 0.8$ | $132.9 \pm 0.8$ | $132.9 \pm 0.8$ | $132.9 \pm 0.8$ |

The results, summarized in Table 1, demonstrate that ICL requires fewer gradient queries to converge to an $\varepsilon$-accurate Nash equilibrium when the transaction fee $\rho \leq 0.12\%$. This empirical observation aligns with our theoretical prediction in Example 1, which suggests that ICL converges faster when $\rho\,\|\text{abs}\,(\mathbf{M})\| \ll \sqrt{\mu\nu} = 1\%$.

## 7 Conclusions, limitations, and future work

In this work we consider the class of monotone games and present a condition that naturally interpolates between the zero-sum and a non-zero-sum class. We develop an efficient gradient-based approach and show its applicability with several examples motivated from the literature.

There are some limitations of our work: (a) in our complexity there is a $\log^2(\frac{D^2}{\varepsilon})$ dependency rather than a single logarithm dependency, and whether this double logarithm dependency can be removed is an interesting question; and (b) whether lower-bound results can be obtained for the new class also remains an open question.

In addition to the above two theoretical limitations, there are several other interesting directions as well: for example, (a) exploring other applications of regularized matrix games with near-zero-sum payoff matrices is an interesting direction; and (b) in the research of semi-cooperative games where competition dominates, applying our methods in more practical examples is another fruitful direction for future research.

## Acknowledgments and Disclosure of Funding

The authors thank for the helpful discussion with Anton Rodomanov during the initial preparation of this project.

RL and KC acknowledge the support of ERC CoG 863818 (ForM-SMArt) and Austrian Science Fund (FWF) 10.55776/COE12.

## References

[1] Bharat Adsul, Jugal Garg, Ruta Mehta, Milind Sohoni, and Bernhard Von Stengel. Fast algorithms for rank-1 bimatrix games. *Operations Research*, 69(2):613–631, 2021.

[2] Ozan Candogan, Asuman Ozdaglar, and Pablo A Parrilo. Dynamics in near-potential games. *Games and Economic Behavior*, 82:66–90, 2013.

[3] Yair Carmon, Arun Jambulapati, Yujia Jin, and Aaron Sidford. Recapp: Crafting a more efficient catalyst for convex optimization. In *International Conference on Machine Learning*, pages 2658–2685. PMLR, 2022.

[4] Xi Chen, Xiaotie Deng, and Shang-Hua Teng. Settling the complexity of computing two-player Nash equilibria. *Journal of the ACM (JACM)*, 56(3):1–57, 2009.

[5] Yunmei Chen, Guanghui Lan, and Yuyuan Ouyang. Accelerated schemes for a class of variational inequalities. *Mathematical Programming*, 165:113–149, 2017.

[6] Constantinos Daskalakis, Paul W Goldberg, and Christos H Papadimitriou. The complexity of computing a Nash equilibrium. *Communications of the ACM*, 52(2):89–97, 2009.

[7] Joseph Halpern and Nan Rong. Cooperative equilibrium: A solution predicting cooperative play. In *International Joint Conference on Autonomous Agents and Multiagent Systems (AAMAS 2013)*, pages 319–326, 2013.

[8] Aamal Hussain, Francesco Belardinelli, and Georgios Piliouras. Beyond strict competition: approximate convergence of multi-agent q-learning dynamics. In *Proceedings of the Thirty-Second International Joint Conference on Artificial Intelligence*, IJCAI '23, 2023.

[9] Sung-Ha Hwang and Luc Rey-Bellet. Strategic decompositions of normal form games: Zero-sum games and potential games. *Games and Economic Behavior*, 122:370–390, 2020.

[10] Adam Kalai and Ehud Kalai. Cooperation in strategic games revisited. *The Quarterly Journal of Economics*, 128(2):917–966, 2013.

[11] Ehud Kalai and Robert W Rosenthal. Arbitration of two-party disputes under ignorance. *International Journal of Game Theory*, 7:65–72, 1978.

[12] Ravi Kannan and Thorsten Theobald. Games of fixed rank: A hierarchy of bimatrix games. *Economic Theory*, 42:157–173, 2010.

[13] Galina M Korpelevich. The extragradient method for finding saddle points and other problems. *Matecon*, 12:747–756, 1976.

[14] Dmitry Kovalev and Alexander Gasnikov. The first optimal algorithm for smooth and strongly-convex-strongly-concave minimax optimization. *Advances in Neural Information Processing Systems*, 35:14691–14703, 2022.

[15] Guanghui Lan and Yan Li. A novel catalyst scheme for stochastic minimax optimization. *arXiv preprint arXiv:2311.02814*, 2023.

[16] Hongzhou Lin, Julien Mairal, and Zaid Harchaoui. Catalyst acceleration for first-order convex optimization: from theory to practice. *Journal of Machine Learning Research*, 18(212):1–54, 2018.

[17] Tianyi Lin, Chi Jin, and Michael I Jordan. Near-optimal algorithms for minimax optimization. In *Conference on Learning Theory*, pages 2738–2779. PMLR, 2020.

[18] Ruta Mehta. Constant rank two-player games are ppad-hard. *SIAM Journal on Computing*, 47(5):1858–1887, 2018.

[19] Aryan Mokhtari, Asuman E Ozdaglar, and Sarath Pattathil. Convergence rate of o(1/k) for optimistic gradient and extragradient methods in smooth convex-concave saddle point problems. *SIAM Journal on Optimization*, 30(4):3230–3251, 2020.

[20] John Nash. The bargaining problem. *Econometrica*, 18(2):155–162, 1950.

[21] John Nash. Non-cooperative games. *Annals of Mathematics*, 54(2), 1951.

[22] John Nash. Two-person cooperative games. *Econometrica: Journal of the Econometric Society*, pages 128–140, 1953.

[23] Arkadi Nemirovski. Information-based complexity of convex programming. *Lecture notes*, 834, 1995.

[24] Arkadi Nemirovski. Prox-method with rate of convergence o (1/t) for variational inequalities with lipschitz continuous monotone operators and smooth convex-concave saddle point problems. *SIAM Journal on Optimization*, 15(1):229–251, 2004.

[25] A. S. Nemirovsky and D. B. Yudin. *Problem complexity and method efficiency in optimization*. Wiley-Interscience, 1983.

[26] Yurii Nesterov. Smooth minimization of non-smooth functions. *Mathematical programming*, 103:127–152, 2005.

[27] Yurii E. Nesterov. *Introductory Lectures on Convex Optimization - A Basic Course*, volume 87 of *Applied Optimization*. Springer, 2004.

[28] Christos Papadimitriou, Emmanouil-Vasileios Vlatakis-Gkaragkounis, and Manolis Zampetakis. The computational complexity of multi-player concave games and Kakutani fixed points. In *Proceedings of the 24th ACM Conference on Economics and Computation*, pages 1045–1045, 2023.

[29] Leonid Denisovich Popov. A modification of the Arrow-Hurwitz method of search for saddle points. *Mat. Zametki*, 28(5):777–784, 1980.

[30] Howard Raiffa. *Arbitration schemes for generalized two-person games*. University of Michigan, 1952.

[31] R Tyrrell Rockafellar. Monotone operators and the proximal point algorithm. *SIAM journal on control and optimization*, 14(5):877–898, 1976.

[32] J Ben Rosen. Existence and uniqueness of equilibrium points for concave n-person games. *Econometrica: Journal of the Econometric Society*, pages 520–534, 1965.

[33] Reinhard Selten. Bewertung strategischer spiele. *Zeitschrift für die gesamte Staatswissenschaft/Journal of Institutional and Theoretical Economics*, H(2):221–282, 1960.

[34] Maurice Sion. On general minimax theorems. *Pacific Journal of Mathematics*, 8(1):171–176, 03 1958.

[35] John Maynard Smith. *Evolution and the Theory of Games*. Cambridge University Press, 1982.

[36] Kiran K. Thekumparampil, Niao He, and Sewoong Oh. Lifted primal-dual method for bilinearly coupled smooth minimax optimization. In Gustau Camps-Valls, Francisco J. R. Ruiz, and Isabel Valera, editors, *Proceedings of The 25th International Conference on Artificial Intelligence and Statistics*, volume 151 of *Proceedings of Machine Learning Research*, pages 4281–4308. PMLR, 28–30 Mar 2022.

[37] Paul Tseng. On linear convergence of iterative methods for the variational inequality problem. *Journal of Computational and Applied Mathematics*, 60(1-2):237–252, 1995.

[38] J von Neumann. Zur theorie der gesellschaftsspiele. *Mathematische annalen*, 100(1):295–320, 1928.

[39] John von Neumann and Oskar Morgenstern. *Theory of games and economic behavior*. Princeton university press, 1947.

[40] Yuanhao Wang and Jian Li. Improved algorithms for convex-concave minimax optimization. *Advances in Neural Information Processing Systems*, 33:4800–4810, 2020.

[41] Jörgen W Weibull. *Evolutionary game theory*. MIT press, 1997.

[42] Junchi Yang, Siqi Zhang, Negar Kiyavash, and Niao He. A catalyst framework for minimax optimization. *Advances in Neural Information Processing Systems*, 33:5667–5678, 2020.

[43] Georgios N Yannakakis and Julian Togelius. *Artificial intelligence and games*, volume 2. Springer, 2018.

[44] Junyu Zhang, Mingyi Hong, and Shuzhong Zhang. On lower iteration complexity bounds for the convex concave saddle point problems. *Mathematical Programming*, 194(1):901–935, 2022.

## A Discussions of imbalanced conditioning

This work studies monotone games in general, while the particular setting of interest (where acceleration happens) is the near-zero-sum games with imbalanced conditioning. Below, we would like to clarify that the conditioning of the two players (that is, the convexity and concavity modulus of the function $h$ in Assumption 1) can indeed be very different, even in zero-sum or near-zero-sum games.

First, generally speaking, zero-sum (or near-zero-sum) games are not necessarily symmetric, and symmetric games are not necessarily zero-sum. So, the games studied in the paper can naturally be non-symmetric or imbalanced between the players.

Moreover, when the strong convexity and strong concavity come from regularizations, these regularizations in a game can also be very imbalanced. For practical concerns, the regularizers are introduced due to many different factors, including but not limited to generalization risk, sparsity level, or proximal operators. Besides, the geometry of strategy spaces $X \subseteq \mathcal{X}$ and $Y \subseteq \mathcal{Y}$ can be very different between the players (for instance, $\dim \mathcal{X} \neq \dim \mathcal{Y}$). All these factors can result in imbalanced regularizations.

Also, strong convexity or strong concavity may not necessarily arise from adding a separate regularizer, but can result directly from the structure of the general coupling function itself. Many games and minimax optimization problems with general coupling have been studied in literature, beginning with the seminal work of [32]. For instance, we give a simple analytical example: the function $h(x, y) = \frac{\log y}{x}$, defined over $x \in [0.3, 1]$ and $y \in [3, 100]$, is 2-strongly convex-0.0001-strongly concave.

Finally, many interesting problems can be reduced to strongly monotone near-zero-sum games with imbalanced conditioning. For instance, in Section 4.3, when the zero-sum part is $\mu$-strongly convex-concave, the reduced game has modulus $\mu$ and $\frac{\varepsilon}{2D_Y^2}$, which can be very imbalanced.

## B Relations to approximate Nash equilibrium

Indeed, an approximate Nash equilibrium can be obtained from an accurate Nash equilibrium [24]. Below, we include this result for self-consistency.

**Proposition 10** ([24])**.** *In a monotone general-sum game, let $\mathbf{z}^* = (\mathbf{x}^*, \mathbf{y}^*)$ be the Nash equilibrium. Let $\bar{\mathbf{z}} = (\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in X \times Y$ and $\gamma \in (0, \frac{1}{\sqrt{2}L}]$. We have*

$$\begin{aligned}
&\max_{\mathbf{x}\in X,\ \mathbf{y}\in Y} u_1(\mathbf{x}, \hat{\mathbf{y}}) - u_1(\hat{\mathbf{x}}, \hat{\mathbf{y}}) + u_2(\hat{\mathbf{x}}, \mathbf{y}) - u_2(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \\
\leq &\max_{\mathbf{x}\in X,\ \mathbf{y}\in Y} \langle \mathcal{F}(\hat{\mathbf{x}}, \hat{\mathbf{y}}), (\hat{\mathbf{x}}, \hat{\mathbf{y}}) - (\mathbf{x}, \mathbf{y}) \rangle \\
\leq &\frac{2}{\gamma} \sqrt{D_X^2 + D_Y^2}\, \|\bar{\mathbf{z}} - \mathbf{z}^*\| \ ,
\end{aligned}$$

*where $\hat{\mathbf{x}} \overset{\text{def}}{=} \Pi_X \left(\bar{\mathbf{x}} + \gamma \nabla_{\mathbf{x}} u_1(\bar{\mathbf{x}}, \bar{\mathbf{y}})\right)$ and $\hat{\mathbf{y}} \overset{\text{def}}{=} \Pi_Y \left(\bar{\mathbf{y}} + \gamma \nabla_{\mathbf{y}} u_2(\bar{\mathbf{x}}, \bar{\mathbf{y}})\right)$.*[7]

*Proof.* Denote $\mathbf{x}_+ \overset{\text{def}}{=} \Pi_X \left(\bar{\mathbf{x}} + \gamma \nabla_{\mathbf{x}} u_1(\hat{\mathbf{x}}, \hat{\mathbf{y}})\right)$, $\mathbf{y}_+ \overset{\text{def}}{=} \Pi_Y \left(\bar{\mathbf{y}} + \gamma \nabla_{\mathbf{y}} u_2(\hat{\mathbf{x}}, \hat{\mathbf{y}})\right)$, and $\mathbf{z}_+ \overset{\text{def}}{=} (\mathbf{x}_+, \mathbf{y}_+)$. Consider any $\widetilde{\mathbf{x}} \in X$ and $\widetilde{\mathbf{y}} \in Y$. By the assignment of $\hat{\mathbf{x}}$, we have

$$\langle \nabla_{\mathbf{x}} u_1(\bar{\mathbf{x}}, \bar{\mathbf{y}}), \hat{\mathbf{x}} \rangle - \frac{1}{2\gamma} \|\hat{\mathbf{x}} - \bar{\mathbf{x}}\|^2 \geq \langle \nabla_{\mathbf{x}} u_1(\bar{\mathbf{x}}, \bar{\mathbf{y}}), \mathbf{x}_+ \rangle - \frac{1}{2\gamma} \|\mathbf{x}_+ - \bar{\mathbf{x}}\|^2 + \frac{1}{2\gamma} \|\mathbf{x}_+ - \hat{\mathbf{x}}\|^2 \ . \quad (5)$$

By the assignment of $\mathbf{x}_+$, we have

$$\langle \nabla_{\mathbf{x}} u_1(\hat{\mathbf{x}}, \hat{\mathbf{y}}), \mathbf{x}_+ \rangle - \frac{1}{2\gamma} \|\mathbf{x}_+ - \bar{\mathbf{x}}\|^2 \geq \langle \nabla_{\mathbf{x}} u_1(\hat{\mathbf{x}}, \hat{\mathbf{y}}), \widetilde{\mathbf{x}} \rangle - \frac{1}{2\gamma} \|\bar{\mathbf{x}} - \widetilde{\mathbf{x}}\|^2 + \frac{1}{2\gamma} \|\mathbf{x}_+ - \widetilde{\mathbf{x}}\|^2 \ . \quad (6)$$

[7] In a Euclidean space $\mathcal{Q}$, for a non-empty, closed, and convex set $Q \subseteq \mathcal{Q}$ and a vertex $\mathbf{u} \in \mathcal{Q}$, let $\Pi_Q(\mathbf{u})$ denote the projection of $\mathbf{u}$ onto $Q$, that is, $\Pi_Q(\mathbf{u}) \overset{\text{def}}{=} \arg\min_{\mathbf{v}\in Q} \|\mathbf{u} - \mathbf{v}\|$ .

In view of

$$\begin{aligned}
&\langle \nabla_{\mathbf{x}} u_1(\hat{\mathbf{x}}, \hat{\mathbf{y}}), \widetilde{\mathbf{x}} - \hat{\mathbf{x}} \rangle \\
&= \langle \nabla_{\mathbf{x}} u_1(\hat{\mathbf{x}}, \hat{\mathbf{y}}), \mathbf{x}_+ - \hat{\mathbf{x}} \rangle + \langle \nabla_{\mathbf{x}} u_1(\hat{\mathbf{x}}, \hat{\mathbf{y}}), \widetilde{\mathbf{x}} - \mathbf{x}_+ \rangle \\
&\overset{(6)}{\leq} \langle \nabla_{\mathbf{x}} u_1(\hat{\mathbf{x}}, \hat{\mathbf{y}}), \mathbf{x}_+ - \hat{\mathbf{x}} \rangle - \frac{1}{2\gamma} \|\mathbf{x}_+ - \bar{\mathbf{x}}\|^2 + \frac{1}{2\gamma} \|\bar{\mathbf{x}} - \widetilde{\mathbf{x}}\|^2 - \frac{1}{2\gamma} \|\mathbf{x}_+ - \widetilde{\mathbf{x}}\|^2 \\
&= \langle \nabla_{\mathbf{x}} u_1(\hat{\mathbf{x}}, \hat{\mathbf{y}}) - \nabla_{\mathbf{x}} u_1(\bar{\mathbf{x}}, \bar{\mathbf{y}}), \mathbf{x}_+ - \hat{\mathbf{x}} \rangle + \langle \nabla_{\mathbf{x}} u_1(\bar{\mathbf{x}}, \bar{\mathbf{y}}), \mathbf{x}_+ - \hat{\mathbf{x}} \rangle - \frac{1}{2\gamma} \|\mathbf{x}_+ - \bar{\mathbf{x}}\|^2 \\
&\quad + \frac{1}{2\gamma} \|\bar{\mathbf{x}} - \widetilde{\mathbf{x}}\|^2 - \frac{1}{2\gamma} \|\mathbf{x}_+ - \widetilde{\mathbf{x}}\|^2 \\
&\overset{(5)}{\leq} \langle \nabla_{\mathbf{x}} u_1(\hat{\mathbf{x}}, \hat{\mathbf{y}}) - \nabla_{\mathbf{x}} u_1(\bar{\mathbf{x}}, \bar{\mathbf{y}}), \mathbf{x}_+ - \hat{\mathbf{x}} \rangle - \frac{1}{2\gamma} \|\hat{\mathbf{x}} - \bar{\mathbf{x}}\|^2 - \frac{1}{2\gamma} \|\mathbf{x}_+ - \hat{\mathbf{x}}\|^2 \qquad (7) \\
&\quad + \frac{1}{2\gamma} \|\bar{\mathbf{x}} - \widetilde{\mathbf{x}}\|^2 - \frac{1}{2\gamma} \|\mathbf{x}_+ - \widetilde{\mathbf{x}}\|^2 \\
&\leq L \|(\hat{\mathbf{x}}, \hat{\mathbf{y}}) - \bar{\mathbf{z}}\| \cdot \|\mathbf{x}_+ - \hat{\mathbf{x}}\| - \frac{1}{2\gamma} \|\hat{\mathbf{x}} - \bar{\mathbf{x}}\|^2 - \frac{1}{2\gamma} \|\mathbf{x}_+ - \hat{\mathbf{x}}\|^2 \\
&\quad + \frac{1}{2\gamma} \|\bar{\mathbf{x}} - \widetilde{\mathbf{x}}\|^2 - \frac{1}{2\gamma} \|\mathbf{x}_+ - \widetilde{\mathbf{x}}\|^2 \\
&\leq \frac{L}{2\sqrt{2}} \|(\hat{\mathbf{x}}, \hat{\mathbf{y}}) - \bar{\mathbf{z}}\|^2 - \frac{1}{2\gamma} \|\hat{\mathbf{x}} - \bar{\mathbf{x}}\|^2 + \frac{1}{2\gamma} \|\bar{\mathbf{x}} - \widetilde{\mathbf{x}}\|^2 - \frac{1}{2\gamma} \|\mathbf{x}_+ - \widetilde{\mathbf{x}}\|^2
\end{aligned}$$

(where we have used $\gamma \leq \frac{1}{\sqrt{2}L}$ in the last inequality), and similarly,

$$\langle \nabla_{\mathbf{y}} u_2(\hat{\mathbf{x}}, \hat{\mathbf{y}}), \widetilde{\mathbf{y}} - \hat{\mathbf{y}} \rangle \leq \frac{L}{2\sqrt{2}} \|(\hat{\mathbf{x}}, \hat{\mathbf{y}}) - \bar{\mathbf{z}}\|^2 - \frac{1}{2\gamma} \|\hat{\mathbf{y}} - \bar{\mathbf{y}}\|^2 + \frac{1}{2\gamma} \|\bar{\mathbf{y}} - \widetilde{\mathbf{y}}\|^2 - \frac{1}{2\gamma} \|\mathbf{y}_+ - \widetilde{\mathbf{y}}\|^2 \ , \qquad (8)$$

we have

$$\begin{aligned}
&\langle \mathcal{F}(\hat{\mathbf{x}}, \hat{\mathbf{y}}), (\hat{\mathbf{x}}, \hat{\mathbf{y}}) - (\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}) \rangle \\
&\overset{(7)(8)}{=} \left(\frac{L}{\sqrt{2}} - \frac{1}{2\gamma}\right) \|(\hat{\mathbf{x}}, \hat{\mathbf{y}}) - \bar{\mathbf{z}}\|^2 + \frac{1}{2\gamma} \|\bar{\mathbf{z}} - (\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})\|^2 - \frac{1}{2\gamma} \|\mathbf{z}_+ - (\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})\|^2 \qquad (9) \\
&\leq \frac{1}{2\gamma} \|\bar{\mathbf{z}} - (\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})\|^2 - \frac{1}{2\gamma} \|\mathbf{z}_+ - (\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})\|^2 \ ,
\end{aligned}$$

where we have used $\gamma \leq \frac{1}{\sqrt{2}L}$ in the last inequality.

Taking $(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}) := (\mathbf{x}^*, \mathbf{y}^*)$ in Equation (9) for the moment, we get

$$\|\mathbf{z}_+ - \mathbf{z}^*\|^2 \overset{(9)}{\leq} \|\bar{\mathbf{z}} - \mathbf{z}^*\|^2 - 2\gamma \langle \mathcal{F}(\hat{\mathbf{x}}, \hat{\mathbf{y}}), (\hat{\mathbf{x}}, \hat{\mathbf{y}}) - \mathbf{z}^* \rangle \leq \|\bar{\mathbf{z}} - \mathbf{z}^*\|^2 \ . \qquad (10)$$

Finally, in view of

$$\begin{aligned}
&\langle \mathcal{F}(\hat{\mathbf{x}}, \hat{\mathbf{y}}), (\hat{\mathbf{x}}, \hat{\mathbf{y}}) - (\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}) \rangle \\
&\overset{(9)}{\leq} \frac{1}{2\gamma} \|\bar{\mathbf{z}} - (\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})\|^2 - \frac{1}{2\gamma} \|\mathbf{z}_+ - (\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})\|^2 \\
&= \frac{1}{2\gamma} \left( \|\bar{\mathbf{x}} - \widetilde{\mathbf{x}}\|^2 - \|\mathbf{x}_+ - \widetilde{\mathbf{x}}\|^2 + \|\bar{\mathbf{y}} - \widetilde{\mathbf{y}}\|^2 - \|\mathbf{y}_+ - \widetilde{\mathbf{y}}\|^2 \right) \\
&\leq \frac{1}{2\gamma} \left( \|\bar{\mathbf{x}} - \widetilde{\mathbf{x}} + \mathbf{x}_+ - \widetilde{\mathbf{x}}\| \cdot \|\bar{\mathbf{x}} - \mathbf{x}_+\| + \|\bar{\mathbf{y}} - \widetilde{\mathbf{y}} + \mathbf{y}_+ - \widetilde{\mathbf{y}}\| \cdot \|\bar{\mathbf{y}} - \mathbf{y}_+\| \right) \\
&\leq \frac{1}{\gamma} \left( D_X \|\bar{\mathbf{x}} - \mathbf{x}_+\| + D_Y \|\bar{\mathbf{y}} - \mathbf{y}_+\| \right) \qquad (11) \\
&\leq \frac{1}{\gamma} \sqrt{D_X^2 + D_Y^2} \cdot \sqrt{\|\bar{\mathbf{x}} - \mathbf{x}_+\|^2 + \|\bar{\mathbf{y}} - \mathbf{y}_+\|^2} \\
&\leq \frac{1}{\gamma} \sqrt{D_X^2 + D_Y^2} \cdot \sqrt{2 \|\bar{\mathbf{z}} - \mathbf{z}^*\|^2 + 2 \|\mathbf{z}_+ - \mathbf{z}^*\|^2} \\
&\overset{(10)}{\leq} \frac{2}{\gamma} \sqrt{D_X^2 + D_Y^2} \cdot \|\bar{\mathbf{z}} - \mathbf{z}^*\| \ ,
\end{aligned}$$

we have

$$u_1(\widetilde{\mathbf{x}}, \hat{\mathbf{x}}) - u_1(\hat{\mathbf{x}}, \hat{\mathbf{y}}) + u_2(\hat{\mathbf{x}}, \widetilde{\mathbf{y}}) - u_2(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \leq \langle \mathcal{F}(\hat{\mathbf{x}}, \hat{\mathbf{y}}), (\hat{\mathbf{x}}, \hat{\mathbf{y}}) - (\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}) \rangle$$

$$\stackrel{(11)}{\leq} \frac{2}{\gamma} \sqrt{D_X^2 + D_Y^2} \cdot \|\bar{\mathbf{z}} - \mathbf{z}^*\| \ ,$$

and the desired bound follows because $\widetilde{\mathbf{x}}$ and $\widetilde{\mathbf{y}}$ can take arbitrary points in $X$ and $Y$, respectively. □

We also state the following sufficient condition for the accurate Nash equilibrium, which can be used as stopping criterion for the optimization algorithms. Similar results can be found, for instance, in [24, 42].

**Proposition 11** (Stopping criterion [24, 42])**.** *In a monotone general-sum game, let $\mathbf{z}^* = (\mathbf{x}^*, \mathbf{y}^*)$ be the Nash equilibrium. Let $\bar{\mathbf{z}} = (\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in X \times Y$, $\gamma \in (0, \frac{1}{2L}]$, and $\mu = \min\{\mu, \nu\}$. We have*

$$\|\bar{\mathbf{z}} - \mathbf{z}^*\|^2 \leq \left( \frac{4}{\mu^2 \gamma^2} - \frac{2}{\mu \gamma} + 16 \right) \|\mathbf{z}_+ - \bar{\mathbf{z}}\|^2 \ ,$$

*where $\mathbf{z}_+ = \Pi_Z (\bar{\mathbf{z}} - \gamma \mathcal{F}(\hat{\mathbf{z}}))$, in which $\hat{\mathbf{z}} = \Pi_Z (\bar{\mathbf{z}} - \gamma \mathcal{F}(\bar{\mathbf{z}}))$.*

*Proof.* We have

$$\begin{aligned}
&(1 - \frac{\mu\gamma}{2}) \|\bar{\mathbf{z}} - \mathbf{z}^*\|^2 - \left( \frac{2}{\mu\gamma} - 1 \right) \|\mathbf{z}_+ - \bar{\mathbf{z}}\|^2 \\
&\leq \|\mathbf{z}_+ - \mathbf{z}^*\|^2 \\
&\stackrel{(9)}{\leq} \|\bar{\mathbf{z}} - \mathbf{z}^*\|^2 - 2\gamma \langle \mathcal{F}(\hat{\mathbf{z}}), \hat{\mathbf{z}} - \mathbf{z}^* \rangle \\
&\leq \|\bar{\mathbf{z}} - \mathbf{z}^*\|^2 - 2\gamma \langle \mathcal{F}(\hat{\mathbf{z}}) - \mathcal{F}(\mathbf{z}^*), \hat{\mathbf{z}} - \mathbf{z}^* \rangle \\
&\leq \|\bar{\mathbf{z}} - \mathbf{z}^*\|^2 - 2\mu\gamma \|\hat{\mathbf{z}} - \mathbf{z}^*\|^2 \\
&\leq \|\bar{\mathbf{z}} - \mathbf{z}^*\|^2 - \mu\gamma \|\bar{\mathbf{z}} - \mathbf{z}^*\|^2 + 2\mu\gamma \|\hat{\mathbf{z}} - \bar{\mathbf{z}}\|^2 \\
&= (1 - \mu\gamma) \|\bar{\mathbf{z}} - \mathbf{z}^*\|^2 - 2\mu\gamma \|\hat{\mathbf{z}} - \bar{\mathbf{z}}\|^2 + 4\mu\gamma \|\hat{\mathbf{z}} - \bar{\mathbf{z}}\|^2 \\
&\leq (1 - \mu\gamma) \|\bar{\mathbf{z}} - \mathbf{z}^*\|^2 - 2\mu\gamma \|\hat{\mathbf{z}} - \bar{\mathbf{z}}\|^2 + 8\mu\gamma \|\mathbf{z}_+ - \bar{\mathbf{z}}\|^2 + 8\mu\gamma \|\mathbf{z}_+ - \hat{\mathbf{z}}\|^2 \\
&\leq (1 - \mu\gamma) \|\bar{\mathbf{z}} - \mathbf{z}^*\|^2 - 2\mu\gamma \|\hat{\mathbf{z}} - \bar{\mathbf{z}}\|^2 + 8\mu\gamma \|\mathbf{z}_+ - \bar{\mathbf{z}}\|^2 + 8\mu\gamma \|\bar{\mathbf{z}} - \gamma\mathcal{F}(\hat{\mathbf{z}}) - \bar{\mathbf{z}} - \gamma\mathcal{F}(\bar{\mathbf{z}})\|^2 \\
&\leq (1 - \mu\gamma) \|\bar{\mathbf{z}} - \mathbf{z}^*\|^2 - 2\mu\gamma \|\hat{\mathbf{z}} - \bar{\mathbf{z}}\|^2 + 8\mu\gamma \|\mathbf{z}_+ - \bar{\mathbf{z}}\|^2 + 8\mu L^2 \gamma^3 \|\hat{\mathbf{z}} - \bar{\mathbf{z}}\|^2 \\
&\leq (1 - \mu\gamma) \|\bar{\mathbf{z}} - \mathbf{z}^*\|^2 + 8\mu\gamma \|\mathbf{z}_+ - \bar{\mathbf{z}}\|^2 \ ,
\end{aligned}$$

where in the second to last inequality we use $\gamma \leq \frac{1}{2L}$. Finally, the desired bound follows from rearrangement. □

# C Discussions on smoothing techniques

In this section, we present the intuition of most existing algorithms for convex-concave minimax optimization under imbalanced conditioning, and explain why similar idea may not work directly when generalized to monotone near-zero-sum games.

Most of existing algorithms for minimax optimization under imbalanced conditioning are based on some smoothing techniques [26]. In minimax optimization, we have $u_1 + u_2 = 0$. Assume without loss of generality that $\mu \leq \nu$. The function $f(\mathbf{x}) \stackrel{\text{def}}{=} -u_1(\mathbf{x}, \mathbf{y}(\mathbf{x}))$ is $\mu$-strongly convex over $\mathbf{x} \in X$, in which $\mathbf{y}(\mathbf{x}) \stackrel{\text{def}}{=} \arg\max_{\mathbf{y} \in Y} u_2(\mathbf{x}, \mathbf{y})$. At the core of these algorithms, they build a function $\hat{f}_t$ and get an inexact solution $\hat{\mathbf{x}}_{t+1}$ at each iteration $t$:

$$\hat{\mathbf{x}}_{t+1} \approx \operatorname*{arg\,min}_{\mathbf{x} \in X} \left[ \hat{f}_t(\mathbf{x}) \stackrel{\text{def}}{=} f(\mathbf{x}) + \frac{\nu}{2} \|\mathbf{x} - \hat{\mathbf{x}}_t\|^2 \right] . \quad (12)$$

The outer loop is an inexact accelerated proximal point algorithm with $\widetilde{\mathcal{O}}\left(\sqrt{\frac{\nu}{\mu}} \cdot \log\left(\frac{1}{\varepsilon}\right)\right)$ iterations [26, 16, 3], and the inner loop of solving the smoothed Equation (12) can be any method with the number of gradient queries $\widetilde{\mathcal{O}}\left(\frac{L}{\nu} \cdot \log\left(\frac{1}{\varepsilon}\right)\right)$ [37]. So, the total gradient complexity is[8]

$$\widetilde{\mathcal{O}}\left(\sqrt{\frac{\nu}{\mu}} \cdot \log\left(\frac{1}{\varepsilon}\right)\right) \cdot \widetilde{\mathcal{O}}\left(\frac{L}{\nu} \cdot \log\left(\frac{1}{\varepsilon}\right)\right) = \widetilde{\mathcal{O}}\left(\frac{L}{\sqrt{\mu\nu}} \cdot \log^2\left(\frac{1}{\varepsilon}\right)\right) .$$

However, if we try to apply the above smoothing techniques to monotone non-zero-sum games, the algorithm may only converge to a Stackelberg solution, which can be very different from the Nash equilibrium in non-zero-sum games.

EXAMPLE 3 (Stackelberg solution). *Consider the case where $X = [0,1] \times [1,2] \subseteq \mathbb{R}^2$ and $Y = [-1,0] \subseteq \mathbb{R}$. Let Player 1 maximize*

$$u_1(\mathbf{x}, y) = -\frac{1}{2}(x_1 - 1)^2 - \frac{1}{2}(x_2 - 1)^2 + \frac{1}{2}x_1 y$$

*over $\mathbf{x} \in X$, and Player 2 maximize*

$$u_2(\mathbf{x}, y) = \frac{1}{2}x_2 y - (y+1)^2$$

*over $y \in Y$. Then, the minimization of $f(x) = -u_1(\mathbf{x}, y(\mathbf{x}))$ will lead to the Stackelberg solution $(\mathbf{x} = \left(\frac{40}{63}, \frac{68}{63}\right),\ y = -\frac{46}{63})$, which is different from the Nash equilibrium $(\mathbf{x} = \left(\frac{5}{8}, 1\right),\ y = -\frac{3}{4})$.*

Therefore, we are not aware of how the smoothing techniques for convex-concave minimax optimization can be applied in NEPs for non-zero-sum games.

# D Proof details

## D.1 Proofs for the results in Section 4.1

*Proof of Proposition 3.* For any $\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in X \times Y$,

$$\varDelta(\mathbf{z}) \geq g(\mathbf{z}) - g(\mathbf{z}) + h(\mathbf{x}, \mathbf{y}) - h(\mathbf{x}, \mathbf{y}) = 0\,,$$

and for all $\widetilde{\mathbf{z}} = (\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}) \in X \times Y$, we have

$$\begin{aligned}
\varDelta(\mathbf{z}) &\geq \frac{1}{2}\left[g(\mathbf{z}) - g(\mathbf{x}, \widetilde{\mathbf{y}}) + h(\mathbf{x}, \widetilde{\mathbf{y}}) - h(\mathbf{x}, \mathbf{y})\right] + \frac{1}{2}\left[g(\mathbf{z}) - g(\widetilde{\mathbf{x}}, \mathbf{y}) + h(\mathbf{x}, \mathbf{y}) - h(\widetilde{\mathbf{x}}, \mathbf{y})\right] \\
&= \frac{1}{2}\left[2g(\mathbf{z}) + u_2(\mathbf{x}, \widetilde{\mathbf{y}}) + u_1(\widetilde{\mathbf{x}}, \mathbf{y})\right] \\
&= \frac{1}{2}\left[u_1(\widetilde{\mathbf{x}}, \mathbf{y}) - u_1(\mathbf{x}, \mathbf{y}) + u_2(\mathbf{x}, \widetilde{\mathbf{y}}) - u_2(\mathbf{x}, \mathbf{y})\right] .
\end{aligned}$$

□

*Proof of Proposition 4.* The (if) part follows directly from Proposition 3. Now we prove the (only if) part. Suppose $\mathbf{z}^* = (\mathbf{x}^*, \mathbf{y}^*)$ is the Nash equilibrium. For all $\widetilde{\mathbf{z}} = (\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}) \in X \times Y$,

$$g(\mathbf{z}^*) - g(\widetilde{\mathbf{z}}) + h(\mathbf{x}^*, \widetilde{\mathbf{y}}) - h(\widetilde{\mathbf{x}}, \mathbf{y}^*) \leq \langle \nabla g(\mathbf{z}^*), \mathbf{z}^* - \widetilde{\mathbf{z}} \rangle + \langle \mathcal{H}(\mathbf{z}^*), \mathbf{z}^* - \widetilde{\mathbf{z}} \rangle \leq 0\,,$$

where in the first inequality we use Assumptions 1 and 2. Then, we have $\varDelta(\mathbf{z}^*) = 0$. □

## D.2 Proofs for the results in Section 4.2

The main technical work in the convergence analysis is to use the properties of our potential function and prove the descent lemma (Lemma 5).

[8]The double logarithm term may be avoided by combining this algorithmic idea with some complicated techniques [14, 3], which we omit here for the simplicity of presentation.

*Proof of Lemma 5.* By Assumption 1, we can upper bound the convex-concave zero-sum part

$$\begin{aligned}h(\mathbf{x}_{t+1},\mathbf{y}^*)-h(\mathbf{x}^*,\mathbf{y}_{t+1})&=h(\mathbf{x}_{t+1},\mathbf{y}_{t+1})-h(\mathbf{x}^*,\mathbf{y}_{t+1})+h(\mathbf{x}_{t+1},\mathbf{y}^*)-h(\mathbf{x}_{t+1},\mathbf{y}_{t+1})\\&\le\langle\nabla_{\mathbf{x}}h(\mathbf{x}_{t+1},\mathbf{y}_{t+1}),\mathbf{x}_{t+1}-\mathbf{x}^*\rangle-\frac{\mu}{2}\|\mathbf{x}_{t+1}-\mathbf{x}^*\|^2\\&\quad-\langle\nabla_{\mathbf{y}}h(\mathbf{x}_{t+1},\mathbf{y}_{t+1}),\mathbf{y}_{t+1}-\mathbf{y}^*\rangle-\frac{\nu}{2}\|\mathbf{y}_{t+1}-\mathbf{y}^*\|^2\\&=\langle\mathcal{H}(\mathbf{z}_{t+1}),\mathbf{z}_{t+1}-\mathbf{z}^*\rangle-\frac{\mu}{2}\|\mathbf{x}_{t+1}-\mathbf{x}^*\|^2-\frac{\nu}{2}\|\mathbf{y}_{t+1}-\mathbf{y}^*\|^2\,.\end{aligned}\tag{13}$$

By Assumptions 2 and 3, we can upper bound the jointly convex coupling part

$$\begin{aligned}g(\mathbf{z}_{t+1})-g(\mathbf{z}^*)&=g(\mathbf{z}_{t+1})-g(\mathbf{z}_t)+g(\mathbf{z}_t)-g(\mathbf{z}^*)\\&\le\langle\nabla g(\mathbf{z}_t),\mathbf{z}_{t+1}-\mathbf{z}_t\rangle+\frac{\delta}{2}\|\mathbf{z}_{t+1}-\mathbf{z}_t\|^2+\langle\nabla g(\mathbf{z}_t),\mathbf{z}_t-\mathbf{z}^*\rangle\\&=\langle\nabla g(\mathbf{z}_t),\mathbf{z}_{t+1}-\mathbf{z}^*\rangle+\frac{\delta}{2}\|\mathbf{z}_{t+1}-\mathbf{z}_t\|^2\,.\end{aligned}\tag{14}$$

In view of

$$\frac{1}{2}\langle\mathbf{z}_{t+1}-\mathbf{z}_t,\mathbf{z}_{t+1}-\mathbf{z}^*\rangle=\|\mathbf{z}_{t+1}-\mathbf{z}^*\|^2-\|\mathbf{z}_t-\mathbf{z}^*\|^2+\|\mathbf{z}_{t+1}-\mathbf{z}_t\|^2\,,\tag{15}$$

and

$$\begin{aligned}&\left\langle\nabla g(\mathbf{z}_t)+\mathcal{H}(\mathbf{z}_{t+1})+\frac{1}{\eta_t}(\mathbf{z}_{t+1}-\mathbf{z}_t),\mathbf{z}_{t+1}-\mathbf{z}^*\right\rangle\\&\le\langle\nabla_{\mathbf{x}}\varphi_t(\mathbf{z}_{t+1}),\mathbf{x}_{t+1}-\mathbf{x}^*\rangle-\langle\nabla_{\mathbf{y}}\varphi_t(\mathbf{z}_{t+1}),\mathbf{y}_{t+1}-\mathbf{y}^*\rangle\\&\overset{(1)}{\le}\varepsilon_t\,,\end{aligned}\tag{16}$$

we have

$$\begin{aligned}0&=-\mathit{\Delta}(\mathbf{z}^*)\le g(\mathbf{z}_{t+1})-g(\mathbf{z}^*)+h(\mathbf{x}_{t+1},\mathbf{y}^*)-h(\mathbf{x}^*,\mathbf{y}_{t+1})\\&\overset{(13)(14)}{\le}\langle\nabla g(\mathbf{z}_t)+\mathcal{H}(\mathbf{z}_{t+1}),\mathbf{z}_{t+1}-\mathbf{z}^*\rangle-\frac{\mu}{2}\|\mathbf{x}_{t+1}-\mathbf{x}^*\|^2-\frac{\nu}{2}\|\mathbf{y}_{t+1}-\mathbf{y}^*\|^2+\frac{\delta}{2}\|\mathbf{z}_{t+1}-\mathbf{z}_t\|^2\\&=\left\langle\nabla g(\mathbf{z}_t)+\mathcal{H}(\mathbf{z}_{t+1})+\frac{1}{\eta_t}(\mathbf{z}_{t+1}-\mathbf{z}_t),\mathbf{z}_{t+1}-\mathbf{z}^*\right\rangle-\frac{1}{\eta_t}\langle\mathbf{z}_{t+1}-\mathbf{z}_t,\mathbf{z}_{t+1}-\mathbf{z}^*\rangle\\&\quad-\frac{\mu}{2}\|\mathbf{x}_{t+1}-\mathbf{x}^*\|^2-\frac{\nu}{2}\|\mathbf{y}_{t+1}-\mathbf{y}^*\|^2+\frac{\delta}{2}\|\mathbf{z}_{t+1}-\mathbf{z}_t\|^2\\&\overset{(15)(16)}{\le}\varepsilon_t+\frac{1}{2\eta_t}\|\mathbf{x}_t-\mathbf{x}^*\|^2-\left(\frac{1}{2\eta_t}+\frac{\mu}{2}\right)\|\mathbf{x}_{t+1}-\mathbf{x}^*\|^2\\&\quad+\frac{1}{2\eta_t}\|\mathbf{y}_t-\mathbf{y}^*\|^2-\left(\frac{1}{2\eta_t}+\frac{\nu}{2}\right)\|\mathbf{y}_{t+1}-\mathbf{y}^*\|^2-\left(\frac{1}{2\eta_t}-\frac{\delta}{2}\right)\|\mathbf{z}_{t+1}-\mathbf{z}_t\|^2\,,\end{aligned}$$

where the first equality follows from Proposition 4 and the first inequality follows from the definition of $\mathit{\Delta}(\cdot)$. Finally, the desired bound follows from $\eta_t\le\frac{1}{\delta}$. □

With Lemma 5, we are ready to prove the complexity of the outer loop (Lemma 6).

*Proof of Lemma 6.* For monotone $\delta$-nearly-zero-sum games and $\eta\le\frac{1}{\delta}$, by Lemma 5, for any $k\in[0,t-1]\cap\mathbb{Z}$, we have

$$\|\mathbf{z}_{k+1}-\mathbf{z}^*\|^2\le(1-\theta)\|\mathbf{z}_k-\mathbf{z}^*\|^2+2\eta\varepsilon_k\,.$$

Then, unrolling this recursion (from $k=t-1,t-2,\cdots,$ to 0) yields

$$\begin{aligned}\|\mathbf{z}_t-\mathbf{z}^*\|^2&\le(1-\theta)^t\|\mathbf{z}_0-\mathbf{z}^*\|^2+2\eta\sum_{k=0}^{t-1}(1-\theta)^{t-k-1}\varepsilon_k\\&\le(1-\theta)^t(D_X^2+D_Y^2)+\frac{2\eta}{\theta}\cdot\max_{k\in[0,t-1]\cap\mathbb{Z}}\varepsilon_k\\&\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\\&=\varepsilon\,,\end{aligned}$$

where the last inequality follows from $t \geq \frac{1}{\theta} \log \frac{2(D_X^2 + D_Y^2)}{\varepsilon}$ and $\varepsilon_t \leq \frac{\theta \varepsilon}{4\eta}$. □

Below, we also include the proof of the gradient complexity of the inner loops for completeness. This result of the inner loops is heavily based on the previous results of optimal gradient methods in minimax optimization (see, for instance, [14, 3, 36, 15]).

*Proof of Lemma 7 [14, 3, 36, 15].* Let $\mathbf{z}_{t+1}^* = (\mathbf{x}_{t+1}^*, \mathbf{y}_{t+1}^*) \in X \times Y$ denote the saddle point of $\varphi_t(\cdot, \cdot)$. Denote

$$\bar{\varepsilon}_t = \frac{\varepsilon_t^2}{8L^2 \left(D_X^2 + D_Y^2\right)} .$$

By Proposition 10, an inexact solution in Equation (1) of Algorithm 1 can be obtained from a pair of decisions $\bar{\mathbf{z}}_{t+1} = (\bar{\mathbf{x}}_{t+1}, \bar{\mathbf{y}}_{t+1}) \in X \times Y$ that satisfies $\left\|\bar{\mathbf{z}}_{t+1} - \mathbf{z}_{t+1}^*\right\|^2 \leq \bar{\varepsilon}_t$.

The function $\varphi_t(\cdot, \cdot)$ is $\left(\eta_t^{-1} + \mu\right)$-strongly convex-$\left(\eta_t^{-1} + \nu\right)$-strongly concave and $2L$-smooth, where the $2L$-smoothness follows from $\eta_t \geq \frac{1}{L}$. Hence, by [14, 3, 36, 15], the aforementioned pair of decisions $\bar{\mathbf{z}}_{t+1}$ can be found within

$$\mathcal{O}\left(\frac{L}{\sqrt{\left(\eta_t^{-1} + \mu\right)\left(\eta_t^{-1} + \nu\right)}} \cdot \log\left(\frac{D_X^2 + D_Y^2}{\bar{\varepsilon}_t}\right)\right)$$

gradient queries. Finally, after substituting the $\bar{\varepsilon}_t$, the desired bound follows. □

Finally, we prove Theorem 1, our main theoretical result.

*Proof of Theorem 1.* The overall gradient complexity is given by the multiplication of outer loop iterations (Lemma 6) and inner loop gradient complexity (Lemma 7):

$$\begin{aligned}
&\mathcal{O}\left(\frac{\eta^{-1} + \min\{\mu, \nu\}}{\min\{\mu, \nu\}} \cdot \log \frac{2(D_X^2 + D_Y^2)}{\varepsilon}\right) \cdot \mathcal{O}\left(\frac{L}{\sqrt{\left(\eta^{-1} + \mu\right)\left(\eta^{-1} + \nu\right)}} \cdot \log \frac{L(D_X^2 + D_Y^2)}{\varepsilon_t}\right) \\
&= \mathcal{O}\left(\frac{\delta + \min\{\mu, \nu\}}{\min\{\mu, \nu\}} \cdot \log \frac{D_X^2 + D_Y^2}{\varepsilon}\right) \cdot \mathcal{O}\left(\frac{L}{\sqrt{(\delta + \mu)(\delta + \nu)}} \cdot \log\left(\frac{L(D_X^2 + D_Y^2)}{\min\{\mu, \nu\} \cdot \varepsilon}\right)\right) \\
&= \mathcal{O}\left(\frac{L}{\min\{\mu, \nu\}} \cdot \sqrt{\frac{\delta + \min\{\mu, \nu\}}{\delta + \max\{\mu, \nu\}}} \cdot \log\left(\frac{L\left(D_X^2 + D_Y^2\right)}{\min\{\mu, \nu\} \cdot \varepsilon}\right) \log\left(\frac{D_X^2 + D_Y^2}{\varepsilon}\right)\right) \\
&= \mathcal{O}\left(\left(\frac{L}{\sqrt{\mu\nu}} + \frac{L}{\min\{\mu, \nu\}} \cdot \min\left\{1, \sqrt{\frac{\delta}{\mu + \nu}}\right\}\right) \cdot \log\left(\frac{L\left(D_X^2 + D_Y^2\right)}{\min\{\mu, \nu\} \cdot \varepsilon}\right) \log\left(\frac{D_X^2 + D_Y^2}{\varepsilon}\right)\right) ,
\end{aligned}$$

where the first relation follows from $\eta = \min\left\{\frac{1}{\delta}, \frac{1}{\min\{\mu, \nu\}}\right\}$. □

### D.3 Proofs for the result in Section 4.3

*Proof of Corollary 9.* We consider the reduced game where Player 1 maximizes $\hat{u}_1 = u_1 - \min\left\{\frac{\varepsilon}{4D_X^2}, \frac{L}{2}\right\} \|\mathbf{x}\|^2 + \min\left\{\frac{\varepsilon}{4D_Y^2}, \frac{L}{2}\right\} \|\mathbf{y}\|^2$ over $\mathbf{x} \in X$ and Player 2 maximizes $\hat{u}_2 = u_2 + \min\left\{\frac{\varepsilon}{4D_X^2}, \frac{L}{2}\right\} \|\mathbf{x}\|^2 - \min\left\{\frac{\varepsilon}{4D_Y^2}, \frac{L}{2}\right\} \|\mathbf{y}\|^2$ over $\mathbf{y} \in Y$. Any $\frac{\varepsilon}{2}$-approximate Nash equilibrium of the reduced game is an $\varepsilon$-approximate Nash equilibrium in the original game.

Denote $\hat{g} \triangleq -\frac{1}{2}(\hat{u}_1 + \hat{u}_2)$ and $\hat{h} \triangleq \frac{1}{2}(-\hat{u}_1 + u_2)$. Then, we have $\hat{h} = h + \left\{\frac{\varepsilon}{4D_X^2}, \frac{L}{2}\right\} \|\mathbf{x}\|^2 - \left\{\frac{\varepsilon}{4D_Y^2}, \frac{L}{2}\right\} \|\mathbf{y}\|^2$, which is $2L$-smooth and $\bar{\mu}$-strongly convex-$\bar{\nu}$-strongly concave. We also have $\hat{g} = -\frac{1}{2}(\hat{u}_1 + \hat{u}_2) = -\frac{1}{2}(u_1 + u_2) = g$, which is jointly convex $\delta$-smooth. By Theorem 1, we obtain the number of gradient queries for an $\frac{\varepsilon^2}{32L^2D^2}$-accurate Nash equilibrium in the reduced game:

$$\mathcal{O}\left(\left(\frac{L}{\sqrt{\bar{\mu}\bar{\nu}}} + \frac{L}{\min\{\bar{\mu}, \bar{\nu}\}} \cdot \min\left\{1, \sqrt{\frac{\delta}{\bar{\mu} + \bar{\nu}}}\right\}\right) \cdot \log\left(\frac{L^2D^2}{\min\{\bar{\mu}, \bar{\nu}\} \cdot \varepsilon}\right) \log\left(\frac{LD^2}{\varepsilon}\right)\right) .$$

Finally, following from Proposition 10, we obtain the desired $\frac{\varepsilon}{2}$-approximate Nash equilibrium of the reduced game by taking an extragradient step from the $\frac{\varepsilon^2}{32L^2D^2}$-accurate Nash equilibrium. $\square$

### D.4 Proofs for the results in Section 5

**Proposition 12** (Convex reformulation in bilinear coupling)**.** *For $\beta_1, \beta_2 \geq 0$ and $\mathbf{M} \in R^{m\times n}$ such that $\sqrt{\beta_1\beta_2} \geq \|\mathbf{M}\|$, the function $\widetilde{g}(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ defined as*

$$\widetilde{g}(\mathbf{x}, \mathbf{y}) = \frac{\beta_1}{2}\|\mathbf{x}\|^2 + \langle \mathbf{M}\mathbf{x}, \mathbf{y}\rangle + \frac{\beta_2}{2}\|\mathbf{y}\|^2$$

*is jointly convex.*

*Proof.* The quadratic function $\widetilde{g}(\cdot,\cdot)$ is bounded below: for all $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$,

$$\begin{aligned}\widetilde{g}(\mathbf{x}, \mathbf{y}) &\geq \frac{\beta_1}{2}\|\mathbf{x}\|^2 - \|\mathbf{M}\mathbf{x}\|\|\mathbf{y}\| + \frac{\beta_2}{2}\|\mathbf{y}\|^2 \\ &\geq \frac{\beta_1}{2}\|\mathbf{x}\|^2 - \sqrt{\beta_1\beta_2}\|\mathbf{x}\|\|\mathbf{y}\| + \frac{\beta_2}{2}\|\mathbf{y}\|^2 \\ &\geq 0\,,\end{aligned}$$

where in the first inequality we used the Cauchy-Schwarz inequality. Therefore, $\widetilde{g}(\cdot,\cdot)$ is jointly convex. $\square$

**Proposition 13** (Convex reformulation in general coupling)**.** *For $\beta \geq 0$ and $g : X \times Y \to \mathbb{R}$ such that $g(\cdot,\cdot)$ is $\beta$-smooth, the function $\widetilde{g}(\cdot,\cdot) : X \times Y \to \mathbb{R}$ defined as*

$$\widetilde{g}(\mathbf{x}, \mathbf{y}) = \frac{\beta}{2}\|\mathbf{x}\|^2 + g(\mathbf{x}, \mathbf{y}) + \frac{\beta}{2}\|\mathbf{y}\|^2$$

*is jointly convex.*

*Proof.* For all $\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in X \times Y$ and $\mathbf{z}' = (\mathbf{x}', \mathbf{y}') \in X \times Y$, we have

$$\begin{aligned}\langle \nabla\widetilde{g}(\mathbf{z}') - \nabla\widetilde{g}(\mathbf{z}), \mathbf{z}' - \mathbf{z}\rangle &= \beta\|\mathbf{z}' - \mathbf{z}\|^2 + \langle \nabla g(\mathbf{z}') - \nabla g(\mathbf{z}), \mathbf{z}' - \mathbf{z}\rangle \\ &\geq \beta\|\mathbf{z}' - \mathbf{z}\|^2 - \beta\|\mathbf{z}' - \mathbf{z}\|^2 \\ &= 0\,,\end{aligned}$$

where the first inequality follows from the $\beta$-smoothness of $g(\cdot,\cdot)$. Therefore, the function $\widetilde{g}(\cdot,\cdot)$ is jointly convex [27, Theorem 2.1.3]. $\square$

## E Illustration of matrix games with transaction fees

We give a simple illustration for matrix games with transaction fees. Let the payoff matrices of Player 1 and Player 2 without transaction fees be

$$\mathbf{M} = \begin{bmatrix} 300 & -200 \\ -100 & 400 \end{bmatrix} \quad \text{and} \quad -\mathbf{M} = \begin{bmatrix} -300 & 200 \\ 100 & -400 \end{bmatrix},$$

respectively. Then,

$$\text{abs}\,(\mathbf{M}) = \begin{bmatrix} 300 & 200 \\ 100 & 400 \end{bmatrix}, \quad \mathbf{M}_+ = \begin{bmatrix} 300 & 0 \\ 0 & 400 \end{bmatrix}, \quad \text{and} \quad \mathbf{M}_- = \begin{bmatrix} 0 & 200 \\ 100 & 0 \end{bmatrix}.$$

Let $1\%$ of transaction fees be imposed on every payment. Then, the payoff matrices of Player 1 and Player 2 with transaction fees are

$$\mathbf{A} = \begin{bmatrix} 297 & -200 \\ -100 & 396 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -300 & 198 \\ 99 & -400 \end{bmatrix},$$

respectively. We also draw the following Table 2 for easier comparisons.

Table 2: An illustration of matrix games with transaction fee $\rho = 0.01$.

| | |
|---|---|
| 300/-300 | -200/200 |
| -100/100 | 400/-400 |

| | |
|---|---|
| 297/-300 | -200/198 |
| -100/99 | 396/-400 |

# F More experimental details

## F.1 Implementation details

We generate the sparse matrix $\mathbf{M}$ following the procedures outlined in [24, 26]: (i) The random seeds are set from 0, 111, 222, ..., and 999; (ii) 100000 coordinates of $\mathbf{M}$ are chosen uniformly at random; (iii) each chosen coordinate is assigned a random value independently drawn from a uniform distribution between $[-1, 1]$; (iv) all remaining coordinates are set to $0$.

We implement our ICL method as described in Algorithm 1. The classic OGDA and classic EG methods are implemented as outlined in [29] and [13], respectively. All solvers are initialized at $(\mathbf{x}_0, \mathbf{y}_0) = (\mathbf{1}_n/n, \mathbf{1}_m/m)$, where $\mathbf{1}_k \in \mathbb{R}^k$ denotes the vector of size $k$ where every element in the vector is equal to 1. The setup for ICL is detailed in Theorem 1. The stepsize for OGDA is set to $\frac{1}{2L}$ following [29, 19], and for EG is set to $\frac{1}{\sqrt{2}L}$ following [13, 24]. For the inner loop, the Lifted Primal Dual method [36] is used, with the theoretical setup maintained as specified in [36, Theorem 2].

## F.2 More details of the experiment runs

We conducted our experiments on `e2-highcpu` vCPUs within the `Google Cloud` environment. The memory requirement of our experiments is quite modest, requiring only sufficient RAM for a few $10000 \times 10000$ sparse matrices (that is, about 60 MB). Each independent run completes within about 3 minutes.

We plot the convergence behaviors in Figure 1. Note that for ICL, only iterates within the outer loop are plotted. Figure 1 shows results for a single seed (seed 0), as plotting all seeds in a single figure would introduce excessive visual complexity due to the unaligned $x$-axis representing the counts of gradient queries in the outer loop. Nonetheless, we observed consistent convergence patterns across different seeds: (i) Transaction fee changes have little impact on the convergence of OGDA and EG, but significantly accelerate the convergence of ICL as $\rho$ decreases; (ii) ICL converges fastest when $\rho \leq 0.12\%$; and (iii) OGDA converges fastest when $\rho \geq 0.15\%$.

Finally, we report CPU times of experiment runs to converge to an $\varepsilon$-accurate Nash equilibrium in Table 3, with error bars indicating 2-sigma variations across 10 independent runs using randomly generated matrices. Table 3 shows that ICL achieves the shortest CPU time when $\rho \leq 0.12\%$, while OGDA achieves the shortest CPU time when $\rho \geq 0.15\%$.

Table 3: The CPU times (in seconds) of the algorithms to converge to an $\varepsilon$-accurate Nash equilibrium under various transaction fees. The error bars indicate 2-sigma variations across the independent runs with 10 randomly generated matrices.

| Methods \ Transaction fee $\rho$ | $0.00\%$ | $0.03\%$ | $0.06\%$ | $0.09\%$ | $0.12\%$ | $0.15\%$ | $0.18\%$ |
|---|---|---|---|---|---|---|---|
| ICL (Algorithm 1) | $\mathbf{19 \pm 0}$ | $\mathbf{49 \pm 0}$ | $\mathbf{93 \pm 0}$ | $\mathbf{142 \pm 1}$ | $\mathbf{167 \pm 0}$ | $247 \pm 2$ | $264 \pm 3$ |
| OGDA [29] | $186 \pm 1$ | $185 \pm 1$ | $185 \pm 1$ | $185 \pm 0$ | $185 \pm 1$ | $\mathbf{185 \pm 1}$ | $\mathbf{186 \pm 1}$ |
| EG [13] | $258 \pm 1$ | $256 \pm 2$ | $258 \pm 2$ | $257 \pm 3$ | $257 \pm 2$ | $257 \pm 2$ | $257 \pm 2$ |

## F.3 Additional runs under different parameter setting

In this section, we run additional numerical experiments under different parameter setting. We change the parameter $\nu = 0.01$, and we vary the transaction fee $\delta$ from $\{0.0\%, 0.3\%, \cdots, 1.8\%\}$. We keep the other parameter settings unchanged.

The results, summarized in Table 4, demonstrate that ICL requires fewer gradient queries to converge to an $\varepsilon$-accurate Nash equilibrium when the transaction fee $\rho$ is below $1.2\%$. This empirical observation

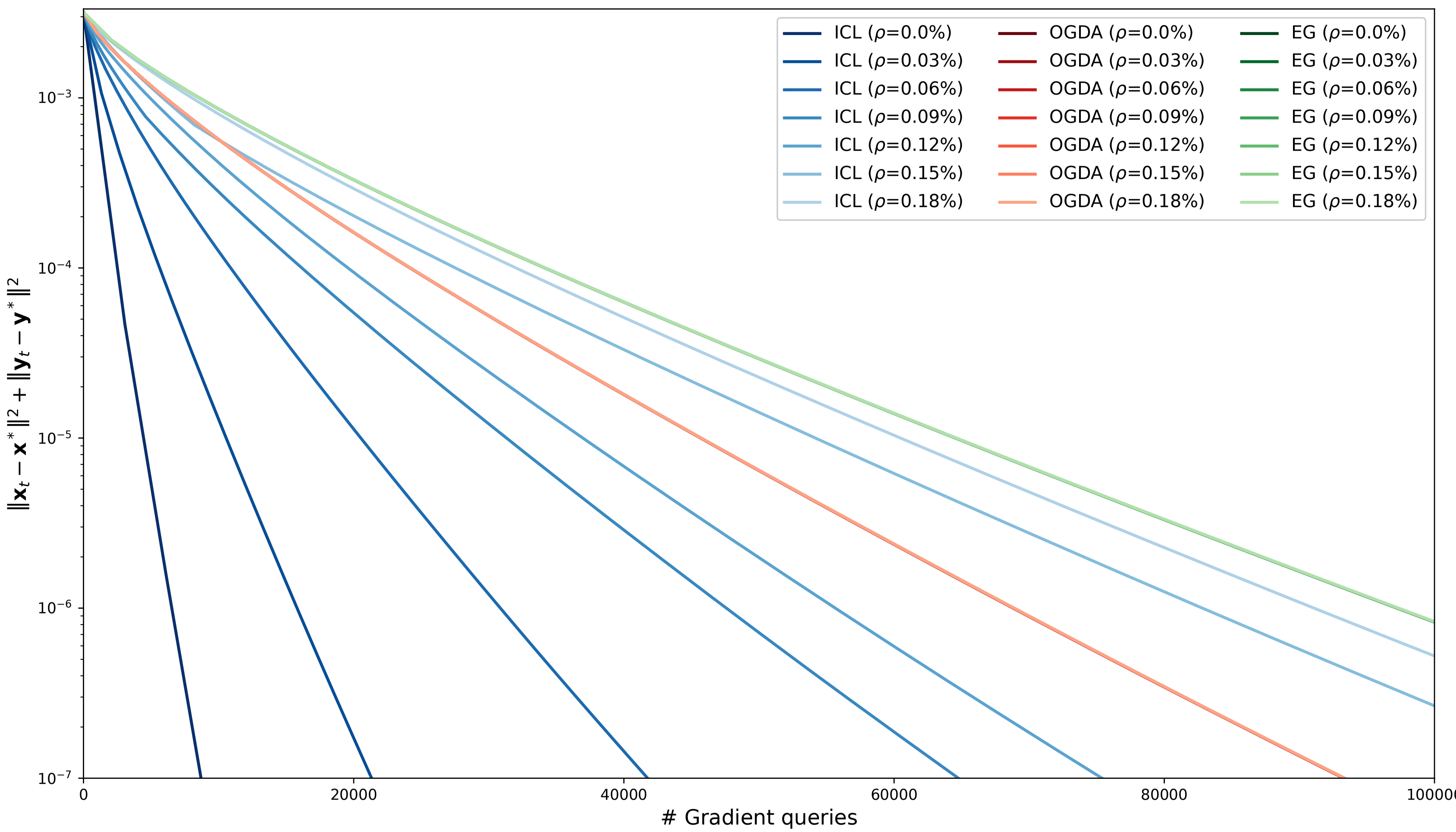


Figure 1: Comparisons of the convergence of the ICL, OGDA, and EG methods with respect to the gradient query counts.

aligns with our theoretical prediction in Example 1, which suggests that ICL converges faster when $\rho \|\text{abs}(\mathbf{M})\| \ll \sqrt{\mu\nu} = 10\%$.

Table 4: Gradient query counts to converge to an $\varepsilon$-accurate Nash equilibrium under various transaction fees. Error bars indicate 2-sigma variations across the independent runs with 10 randomly generated matrices.

| Methods \ Transaction fee $\rho$ | $0.0\%$ | $0.3\%$ | $0.6\%$ | $0.9\%$ | $1.2\%$ | $1.5\%$ | $1.8\%$ |
|---|---|---|---|---|---|---|---|
| ICL (Algorithm 1) | $\mathbf{924 \pm 0}$ | $\mathbf{924 \pm 0}$ | $\mathbf{824 \pm 0}$ | $\mathbf{1030 \pm 0}$ | $\mathbf{1236 \pm 0}$ | $1648 \pm 0$ | $2060 \pm 0$ |
| OGDA [29] | $1364 \pm 0$ | $1364 \pm 0$ | $1364 \pm 0$ | $1361 \pm 4$ | $1359 \pm 5$ | $\mathbf{1353 \pm 6}$ | $\mathbf{1350 \pm 6}$ |
| EG [13] | $1848 \pm 0$ | $1848 \pm 0$ | $1848 \pm 0$ | $1848 \pm 0$ | $1848 \pm 0$ | $1848 \pm 0$ | $1848 \pm 0$ |

We also observe that the CPU times in this setting are within 5 seconds for all independent runs.